GENERAL TOPOLOGY
EXERCISES FOR SESSION 1 (TUE 27.2)

Exercise 1. *Let $X = \mathbb{R}$ and let $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. Prove that $\mathcal{T}$ is a topology.*

Exercise 2. *Find a collection of open sets in $\mathbb{R}^2$ whose intersection is not open.*

Exercise 3. *Let $\mathbb{R}$ be equipped with the discrete topology. Find the boundary of the open interval $(0, 1)$. How does the answer change if we use the mini-topology $\mathcal{T}_{mini} = \{\emptyset, \mathbb{R}\}$ instead?*

Exercise 4. *Let $A \subset X$. Show that the sets $\mathrm{int}\, A$ and $\mathrm{ext}\, A$ are open sets, and that ∂A is a closed set.*

Exercise 5. *Show that if $B \subset A$ is open and $\mathrm{int}\, A \subset B$, then $B = \mathrm{int}\, A$.*

Exercise 6. *Find the interior and boundary of the following sets in the standard topology of $\mathbb{R}^2$:*
   (1) $\{(x, y) : y = 0\}$.
   (2) $\{(x, y) : x > 0 \text{ and } y \neq 0\}$.
   (3) $\{(x, y) : x \in \mathbb{Q}\}$.
   (4) $\{(x, y) : 0 < x^2 + y^2 \leq 1\}$.
   (5) $\{(x, y) : x, y \in \mathbb{Q}, x > y\}$.

Exercise 7. *Does it always hold that $\mathrm{int}(A \cup B) = \mathrm{int}\, A \cup \mathrm{int}\, B$? If not, what about an inclusion $\subset$ in either direction?*

Exercise 8. *Let $A \subset B \subset X$. Suppose that B is closed and A is a closed set in B (with the relative topology). Show that A is closed in X.*

Exercise 9. *Show that the relative topology on $\mathbb{Z}$ as a subset of $\mathbb{R}$ is the discrete topology.*

Exercise 10. *Let $A \subset B \subset X$. We denote by $\mathrm{int}\, A$ the interior of A as a subset of X, and by $\mathrm{int}_B A$ the interior of A in the relative topology on B. Show that $\mathrm{int}\, A \subset \mathrm{int}_B A$.*
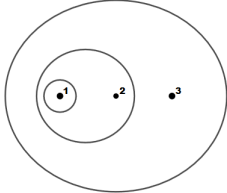
Exercise 11. *Let $X = \mathbb{R}$ and $\mathcal{T}_{co} = \{\mathbb{R}\} \cup \{A \subset \mathbb{R} : \mathbb{R} \setminus A \text{ is countable.}\}$. Is $\mathcal{T}_{co}$ a topology on $\mathbb{R}$?*

Exercise 12. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology. What about $\mathcal{T}_1 \cup \mathcal{T}_2$?*

Exercise 13. *Let $A \subset X$. Prove that*

- $\partial\partial A \subset \partial A$.
- $\partial\partial A = \partial A$ *if A is closed.*
- $\partial\partial\partial A = \partial\partial A$.

Exercise 14. *Let $X = \{1, 2, 3\}$. We have drawn one topology $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, X\}$ for X in the attached picture:*

How many topologies of X exist in total?