GENERAL TOPOLOGY
EXERCISES FOR SESSION 5 (TUE 12.3)

Exercise 1. *Prove Theorem 2.10: If the topology on X is induced by a family of maps $f_i : X \to Y_i, i \in I$, then a sequence $(x_n) \subset X$ converges to a limit x if and only if $f_i(x_n) \to f_i(x)$ for all $i \in I$.*

Exercise 2. *Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.*

Exercise 3. *Use the previous exercise to directly prove a statement from the previous exercise set: If $f, g : X \to Y$ are continuous and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is closed.*

Exercise 4. *Let X, Y be Hausdorff. Prove that $X \times Y$ is Hausdorff. Does the same proof work for arbitrary product spaces?*

Exercise 5. *Let us induce a topology $\mathcal{T}$ on $\mathbb{R}^2$ using the map $f : \mathbb{R}^2 \to \mathbb{R}$ given as $f(x, y) = \sin(x + y)$. Find the closure of $\{(0, 0)\}$ under this topology (draw a picture).*

Exercise 6. *Show that for $A \subset X$ and $B \subset Y$ we have $\overline{A \times B} = \bar{A} \times \bar{B}$ in the product topology of $X \times Y$.*

Exercise 7. *Prove that an injective map $f : X \to Y$ is an embedding if and only if f induces the topology on X.*

Exercise 8. *Let $f : X \to Y$ be continuous. Show that the graph $G_f = \{(x, f(x)) : x \in X\} \subset X \times Y$ is homeomorphic with X.*

Exercise 9. *Show that if $f : X \to Y$ and X has the topology induced by f, then for any $A \subset X$ we have $\bar{A} = f^{-1}(\overline{f(A)})$.*

Exercise 10. *Show that the topology of pointwise convergence on $\mathcal{F}(X, Y)$ is just the topology induced by the collection of maps $\chi_x : \mathcal{F}(X, Y) \to Y$ defined by $\chi_x(f) = f(x)$.*