

GENERAL TOPOLOGY
EXERCISES FOR SESSION 11 (TUE 9.4)

X is a topological vector space in all of the exercises.

Exercise 1. Let $x_0 \in X$. Show that the single translation map $T_{x_0} : X \rightarrow X$ given by $T_{x_0}(x) = x + x_0$ is continuous (and also a homeomorphism).

Exercise 2. Prove that if $U \subset X$ is open, then $U + A$ is open for any $A \subset X$.

Exercise 3. Prove that if $A, B \subset X$, then $\overline{A} + \overline{B} \subset \overline{A + B}$.

Exercise 4. Give an example of closed sets $A, B \subset \mathbb{R}^2$ such that $A + B$ is not closed.

Exercise 5. A subset $A \subset X$ is called **balanced** if $\lambda A \subset A$ whenever $|\lambda| \leq 1$. Prove that for every neighbourhood U of 0, there is a balanced neighbourhood V of 0 such that $V \subset U$.

Hint: Use the continuity of scalar multiplication $\mathcal{M} : \mathbb{R} \times X \rightarrow X$.

Exercise 6. Prove that $\overline{\{0\}}$ is a linear subspace of X .

Exercise 7. We say that a subset $A \subset X$ is **bounded** (in the topological vector space sense) if, for every neighbourhood V of 0, there is $\lambda > 0$ such that $A \subset \lambda V$.

Prove that every compact subset of X is bounded.

Exercise 8. Let $F : X \rightarrow \mathbb{R}$ be a continuous linear functional. Show that if $A \subset X$ is bounded (in the topological vector space sense), then $F(A)$ is bounded in \mathbb{R} (in the usual sense).

Exercise 9. Let $T : X \rightarrow Y$ be a linear function between two topological vector spaces. Show that T is continuous if and only if it is continuous at 0.

Exercise 10. Let X be a normed space, in which case X^* is also a topological vector space with the topology given by the operator norm $\|F\| := \sup_{\|x\| \leq 1} |F(x)|$. Which topology is finer on X^* , the weak*-topology or the weak topology (of X^* , not X)?