GENERAL TOPOLOGY
HOMEWORK FOR WEEK 5

DEADLINE: MON 8.4, 23:59. **NOTE: 2 WEEK DEADLINE**

Some hints are on the next page.

Exercise 1. *Let X be separable and $f : X \to Y$ be a continuous surjection. Prove that Y is separable.*

Exercise 2. *Prove that a compact metrizable space is second-countable.*

Exercise 3. *Let X have two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1 \subset \mathcal{T}_2$. Suppose that $\mathcal{T}_1$ is Hausdorff and $\mathcal{T}_2$ is compact. Prove that $\mathcal{T}_1 = \mathcal{T}_2$.*

Exercise 4. *Let X be locally compact, Hausdorff, and second-countable. Prove that there exists a sequence of open sets $U_1, U_2, \dots$ such that all of the following hold:*

- $\overline{U_j}$ *is compact for all* j.
- $\overline{U_j} \subset U_{j+1}$ *for all* j.
- $X = \bigcup_j U_j$.

Hint for Exercise 1: This one is straightforward. Good luck!

Hint for Exercise 2: For each n, cover the space by balls of radius $1/n$.

Hint for Exercise 3: Two ways:
a) Show that closed sets in $\mathcal{T}_2$ are in $\mathcal{T}_1$
b) Fancy way: Exercise 2 from Session 6

Hint for Exercise 4: Theorem 4.4 in the lecture notes is useful here. Prove that if $\mathcal{B}$ is your countable basis, then $\mathcal{B}' = \{B \in \mathcal{B} : \bar{B} \text{ is compact}\}$ is still a basis. Find the sets U_j inductively using the elements in $\mathcal{B}'$.