GENERAL TOPOLOGY
EXERCISES FOR SESSION 2 (WED 28.2)

Exercise 1. *Prove that* $\bar{A} = \bigcap \{F : A \subset F \text{ and } F \text{ is closed}\}$.

Exercise 2. *Determine the closure of the set* $\{0\}$ *in the topology of* $\mathbb{R}$ *given by* $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$.

Exercise 3. *Prove Theorem 1.3 in the lecture notes.*

Exercise 4. *Provide an example where* $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Exercise 5. *Prove that* $\bar{A} = A \cup \operatorname{acc} A$.

Exercise 6. *Prove Theorem 1.4 in the lecture notes.*

Exercise 7. *Let* $A, B \subset X$ *be two subsets. What is the coarsest topology on* X *which contains both the sets* A *and* B*?*

Exercise 8. *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be bases for two topologies* $\mathcal{T}$ *and* $\mathcal{T}'$ *on* X. *Prove that* $\mathcal{T}'$ *is finer than* $\mathcal{T}$ *if and only if the following holds:*

For each $x \in X$ *and each* $B \in \mathcal{B}$ *such that* $x \in B$, *there is a* $B' \in \mathcal{B}'$ *such that* $x \in B' \subset B$.

Exercise 9. *Show that* $\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{Q}\}$ *provides a basis for the standard topology on* $\mathbb{R}$.

Exercise 10. *Suppose that* X *is an infinite set with a topology in which every infinite set is open. Show that* X *has the discrete topology.*

Exercise 11. *Given a collection* $\mathcal{S}$ *of subsets of* X. *Let*

$$\mathcal{T} = \bigcap \{\mathcal{T}' : \mathcal{S} \subset \mathcal{T}' \text{ and } \mathcal{T}' \text{ is a topology of } X\}.$$

Check that this defines a topology $\mathcal{T}$ *on* X.

Exercise 12. *Let us consider the collection of open cones pointing towards the right in* $\mathbb{R}^2$*:*

$$C_{x_0, y_0, k} := \{(x, y) \in \mathbb{R}^2 : |y - y_0| < k(x - x_0)\}, \qquad x_0, y_0 \in \mathbb{R}, k > 0.$$

Show that these cones are a basis for a topology on $\mathbb{R}^2$ *(a visual explanation is fine).*

Find the closure of some sets in this topology: the origin $(0, 0)$, *the* y*-axis* $\{(x, y) : x = 0\}$, *and the upper half plane* $\{(x, y) : y > 0\}$.