Matrix-valued A_p weights

For the entirety of this essay we will assume that $1 < p < \infty$ and $n, d \in \mathbb{N}_{\geq 1}$. We write $A \lesssim B$, if $A \leq CB$ for some fixed constant $0 < C < \infty$. We also write $A \eqsim B$, if $B \lesssim A \lesssim B$. Furthermore, we denote $A \lesssim_{p_1,\cdots,p_n} B$, when we allow the implicit constant to depend on the parameters $\{p_1, \cdots, p_n\}$.

A matrix-valued function $W \colon \mathbb{R}^d \to \mathbb{C}^{n \times n}$ is a weight, if it is self-adjoint and almost everywhere positive definite. In this essay we will give two definitions for the matrix-valued analogue of the Muckenhoupt A_p weight class and prove that they are equivalent. We will also compare some known matrix weight results with the scalar-valued analogues.

In the scalar-valued case the A_p condition implies boundedness for many operators in the weighted L^p space. In fact, boundedness of the Hilbert transform in the weighted space gives an equivalent condition. Thus it seems reasonable that the matrix A_p condition should be equivalent to the vector valued weighted norm inequality. We extend an operator that is operating on scalar-valued functions to the vector valued setting by taking the mapping component-wise. Let W be a matrix weight and $\vec{f}$ be a vector-valued function on $\mathbb{R}^d$. Then we say that $\vec{f} \in L^p(W)$, if

$$\|f\|_{L^p(W)} := \left(\int_{\mathbb{R}^d} |W^{\frac{1}{p}}(x)\vec{f}(x)|^p \right)^{\frac{1}{p}} < \infty.$$

In [TV] it was shown that the boundedness of the Hilbert transform on $L^2(W)$ is equivalent to

$$\sup_Q \left| \left(\fint_Q W \right)^{\frac{1}{2}} \left(\fint_Q W^{-1} \right)^{\frac{1}{2}} \right|_{op} < \infty, \tag{1}$$

where the supremum is taken over all cubes.

In order to generalize this for other values of p, we consider norms ρ_t on $\mathbb{R}^n$ defined by

$$\rho_t(x) := |W^{\frac{1}{p}}(t)x|,$$

where $t \in \mathbb{R}^d$. Recall that the dual norm of a norm ρ is defined by

$$\rho^*(x) := \sup_{y \neq 0} \frac{|\langle x, y \rangle|}{\rho(y)}.$$

The dual norm of ρ_t is

$$\rho_t^*(x) \coloneqq \sup_{y\neq 0} \frac{|\langle x, y\rangle|}{|W^{\frac{1}{p}}(t)y|} = |W^{-\frac{1}{p}}(t)x|.$$

Averaging these norms over a cube $Q \subset \mathbb{R}^d$ yields new norms

$$\rho_{p,Q}(x) \coloneqq \left(\fint_Q \rho_t(x)^p \,\mathrm{d}t\right)^{\frac{1}{p}} \qquad \text{and} \qquad \rho_{p',Q}^*(x) \coloneqq \left(\fint_Q \rho_t^*(x)^{p'} \,\mathrm{d}t\right)^{\frac{1}{p'}}.$$

The condition (1) was generalized in [NT] where it was shown that the boundedness of the Hilbert transform on $L^p(W)$ is equivalent to the following definition.

Definition 2. Let $1 < p < \infty$. A weight matrix W is a matrix A_p weight, if

$$\rho_{p',Q}^* \leq C\left(\rho_{p,Q}\right)^*,$$

for every cube $Q \subset \mathbb{R}^d$.

Remark 3. Since $\rho_t^*(x) = |W^{-\frac{1}{p}}(t)x| = |(W^{-\frac{p'}{p}})^{\frac{1}{p'}}(t)x|$ and similarly $\rho_t(x) = |(W^{-\frac{p'}{p}})^{-\frac{1}{p'}}(t)x|$, we get that $W \in A_p$ is equivalent with $W^{-\frac{p'}{p}} \in A_{p'}$.

Remark 4. The reverse inequality is true with constant 1. Indeed, by the definition of dual norms and Hölder's inequality we have

$$|\langle x, y\rangle| = \fint_Q |\langle x, y\rangle| \,\mathrm{d}t \leq \fint_Q \rho_t^*(x)\rho_t(y) \,\mathrm{d}t \leq \rho_{p',Q}^*(x)\rho_{p,Q}(y).$$

Then we can move the terms around and take supremums.

Remark 5. In the case $p = 2$, the above definition reduces to (1). To see this fix a cube $Q \subset \mathbb{R}^d$ and note that

$$\begin{aligned}
\rho_{2,Q}(x) = \left(\fint_Q |W^{\frac{1}{2}}(t)x|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} &= \left(\fint_Q \langle W(t)x, x\rangle \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= \left\langle \left(\fint_Q W\right) x, x \right\rangle^{\frac{1}{2}} = \left|\left(\fint_Q W\right)^{\frac{1}{2}} x\right|
\end{aligned}$$

and hence

$$(\rho_{2,Q})^*(y) = \left|\left(\fint_Q W\right)^{-\frac{1}{2}} y\right|.$$

By similar calculations we also have

$$\rho_{2,Q}^*(y) = \left|\left(\fint_Q W^{-1}\right)^{\frac{1}{2}} y\right|.$$

Thus taking supremums over $y \neq 0$ from both sides of the inequality $\frac{\rho_{2,Q}^*}{(\rho_{2,Q})^*} \leq C$ yields

$$\left|\left(\fint_Q W\right)^{\frac{1}{2}} \left(\fint_Q W^{-1}\right)^{\frac{1}{2}}\right|_{op} \leq C.$$

Sometimes it is useful to work with an equivalent A_p condition that avoids the norms ρ and can be stated completely using only the matrices W. This condition was introduced in [R].

Definition 6. Let $1 < p < \infty$. A weight matrix W is a matrix A_p weight, if

$$\fint_Q \left(\fint_Q \left| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right|_{op}^{p'} \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}x \leq C,$$

for every cube $Q \subset \mathbb{R}^d$.

Theorem 7. *Definitions 2 and 6 are equivalent.*

The following classical result from convex geometry is essential in the proof of the above theorem.

Theorem 8 (John ellipsoid theorem [J])**.** *Let $K \subset \mathbb{R}^n$ be a compact convex symmetric set. Then there is a closed ellipsoid E centred at the origin such that $E \subset K \subset \sqrt{n}E$.*

If ρ is a norm on $\mathbb{R}^n$, then there exists a positive definite self-adjoint matrix A such that $\rho(x) \eqsim_n |Ax|$. To see this, consider the John ellipsoid E of the ρ-unit ball $\{x : \rho(x) \leq 1\}$. Then choosing A to be the positive definite matrix that maps E to the standard unit ball of $\mathbb{R}^n$, then the John ellipsoid theorem gives

$$\{x : |Ax| \leq 1\} = E \subset \{x : \rho(x) \leq 1\} \subset \sqrt{n}E = \{x : |Ax| \leq \sqrt{n}\}.$$

The matrix A is called the reducing matrix of ρ.

We also need the following useful norm lemma.

Lemma 9. *Let $(e_i)_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$. Then for any matrix W and any $p > 0$, we have*

$$|W|_{op}^p \eqsim_{n,p} \sum_{i=1}^n |We_i|^p.$$

Proof. Denoting $x_i \coloneqq \langle x, e_i \rangle$ we have

$$|W|_{op}^p = \sup_{|x| \leq 1} \left| W \sum_{i=1}^n x_i e_i \right|^p \lesssim_p \sup_{|x| \leq 1} |x_i|^p \sum_{i=1}^n |We_i|^p \leq \sum_{i=1}^n |We_i|^p$$

and conversely

$$\sum_{i=1}^n |We_i|^p \lesssim_n \max_{i=1,\dots,n} |We_i|^p \leq |W|_{op}^p.$$

$\square$

Now we are ready to prove theorem 7.

Proof of theorem 7. Let A_Q be the reducing matrix of $\rho_{p,Q}$ and B_Q be the reducing matrix of $\rho^*_{p',Q}$. Then by taking supremums over $y \neq 0$ from both sides of the inequality

$$\frac{\rho^*_{p',Q}}{(\rho_{p,Q})^*} \leq C$$

we see that the inequality in definition 2 is equivalent with $|A_Q B_Q|_{op} \leq C$. Thus it suffices to prove that

$$\fint_Q \left(\fint_Q \left| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right|_{op}^{p'} \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}x \eqsim_{n,p} |A_Q B_Q|_{op}^p.$$

Note that since $W^{\frac{1}{p}}$ and $W^{-\frac{1}{p}}$ are self-adjoint, we have

$$\left| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right|_{op} = \left| W^{-\frac{1}{p}}(y) W^{\frac{1}{p}}(x) \right|_{op}.$$

Applying this with the norm lemma yields

$$\begin{aligned}
&\fint_Q \left(\fint_Q \left| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right|_{op}^{p'} \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}x \\
&\qquad \eqsim_{n,p} \fint_Q \left(\fint_Q \sum_{i=1}^n \left| W^{-\frac{1}{p}}(y) W^{\frac{1}{p}}(x) e_i \right|^{p'} \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}x \\
&\qquad \eqsim_{n,p} \sum_{i=1}^n \fint_Q \left(\fint_Q \left| W^{-\frac{1}{p}}(y) W^{\frac{1}{p}}(x) e_i \right|^{p'} \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}x \\
&\qquad = \sum_{i=1}^n \fint_Q \rho^*_{p',Q} \left(W^{\frac{1}{p}}(x) e_i \right)^p \mathrm{d}x.
\end{aligned}$$

The last expression is comparable with

$$\begin{aligned}
\sum_{i=1}^n \fint_Q \left| B_Q W^{\frac{1}{p}}(x) e_i \right|^p \mathrm{d}x &\eqsim_{n,p} \sum_{i=1}^n \fint_Q \left| W^{\frac{1}{p}}(x) B_Q e_i \right|^p \mathrm{d}x \\
&= \sum_{i=1}^n \rho_{p,Q} (B_Q e_i)^p \mathrm{d}x \\
&\eqsim_{n,p} \sum_{i=1}^n |A_Q B_Q e_i|^p \mathrm{d}x \\
&\eqsim_{n,p} |A_Q B_Q|_{op}^p .
\end{aligned}$$

□

Note that if we assume that W is a scalar weight, then the sharp constant form definition 6 is equal to the scalar A_p characteristic. Thus it is natural to define $[W]_{A_p}$ to be the sharp constant C from definition 6.

It is well known that in the scalar case the A_p condition implies that a general Calderón-Zygmund operator is bounded in the weighted L^p space. In fact, in [H] the following sharp quantitative weighted bound was proven to be true.

Theorem 10 (A_2 theorem)**.** *Let T be a CZO and $w \in A_2$. Then*

$$\|Tf\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)},$$

where C depends only on T and d.

The dependence on the weight characteristic $[w]_{A_2}$ is sharp in the above theorem. In [DGPP] it was proven that this linear A_2 bound implies the following sharp weighted inequality for all $p > 1$.

Theorem 11. *Let T be a CZO and $1 \leq p < \infty$. Then*

$$\|Tf\|_{L^p(w)} \leq C[w]_{A_p}^{\alpha}\|f\|_{L^p(w)},$$

where $\alpha = \max\{1, p' - 1\}$ and C dependes only on T, d and p.

In the matrix case the qualitative version is also true. In fact, in [NPTV] the quantitative A_2 bound

$$\|T\vec{f}\|_{L^2(W)} \leq C[W]_{A_2}^{\frac{3}{2}}\|\vec{f}\|_{L^2(W)},$$

where C depends only on T and d, was shown to be true. Whether one can obtain a linear bound in the matrix case is an open question. Lastly, a similar extrapolation result holds in the matrix case.

Theorem 12 (Rubio de Francia matrix weighted extrapolation [BC])**.** *Let T be a sublinear operator. Suppose that for some $p_0 \in [1, \infty]$, there exists an increasing function K_{p_0} such that for every $W_0 \in A_{p_0}$, we have*

$$\|T\vec{f}\|_{L^{p_0}(W)} \leq K_{p_0}([W_0]_{A_{p_0}})\|\vec{f}\|_{L^{p_0}(W)}.$$

Then for all $p \in]1, \infty[$, and for all $W \in A_p$, we have

$$|T\vec{f}\|_{L^p(W)} \leq K_p(p, p_0, n, d, [W_0]_{A_p})\|\vec{f}\|_{L^p(W)},$$

where

$$K_p(p, p_0, n, d, [W_0]_{A_p}) = C(p, p_0)K_{p_0}\left(C(p, p_0, n, d)[W]_{A_p}^{\max\{\frac{p}{p_0}, \frac{p'}{p_0'}\}}\right).$$

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