Fourier analysis on LCA groups

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1 Introduction

In the context of analysis, standard Fourier analysis over $\mathbb{R}^n$ is almost always sufficient. However, in more abstract settings, a more general form of Fourier analysis can give great insight. For example, the existence on Fourier series on $L^2[0,1)$ can simply be viewed as the space regular representation of S^1 on $L^2(S^1)$ decomposing into a direct sum of characters.

This essay will give a small approachable exposition into how to develop this generalized theory. General Fourier analysis is best developed using the theory of Banach algebras and in particular, Gelfand theory. However, to keep the text engaging to anyone with a solid understanding of basic real analysis, most algebra is omitted. For a rigorous study of the topic, [1] is an excellent reference.

We will first discuss Haar measures, which give a canonical measure on a locally compact (topological) group G—in the same sense that one might consider the Lebesgue measure canonical on $\mathbb{R}^n$. After that, we will discuss Pontryagin duals: the character groups of locally compact abelian (LCA) groups endowed with the compact-open topology. Finally, we will develop the basic theory on Fourier analysis on LCA groups.

2 Preliminary notions

Defintion 2.1. We call a topological space G endowed with a group operation $\cdot : G \times G \to G$ a *topological group* if

$$G \times G \to G, (x, y) \mapsto x \cdot y^{-1}$$

is continuous. We will always denote the identity element of G by 1.

In the following chapters all groups are assumed to be topological and in particular, ALWAYS Hausdorff, unless the context tells otherwise. This is because the word "compact" is really reserved for Hausdorff spaces where it truly has important meaning. We use the following abbreviations:

- LC: locally compact
- LCA: locally compact abelian
- CA: compact abelian

We will also let

- $C_c(G) = \{f : G \to \mathbb{C} \mid f \text{ is continuous and has compact support}\}$
- $C_0(G) = \{f : G \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\}$

with vanishing at infinity formally meaning that, for every $\varepsilon > 0$, there's a compact $K \subseteq G$ with the function $< \varepsilon$ a.e. on $G \setminus K$.

The author is lazy, so we frequently use notation like $f(g\cdot)$ to mean the function $x \mapsto f(gx)$. Laziness is also why almost everything is written manually without environments. The references are also lazy. Sorry not sorry.

We use the following well-known lemma extensively.

Lemma 2.2. Let G be an LC group and let $f \in C_c(G)$. Now f is both left and right uniformly continuous in the sense that for $g \in G$,

$$\|f(g\cdot) - f\|_\infty \to 0 \text{ and } \|f(\cdot g) - f\|_\infty \to 0$$

as $g \to 1$.

Proof. Proposition 2.6. of [1].

Definition 2.3. We call a Borel measure μ on a Hausdorff space X a *Radon measure* if it is

1. Locally finite: for all compact $K \subset X$ we have $\mu(K) < +\infty$.
2. Inner regular on open sets: for all open sets U we have

   $$\mu(U) = \sup_{K \subset U} \mu(K)$$

   where the supremum is taken over compact K.
3. Outer regular on Borel sets: for all Borel sets E we have

   $$\mu(E) = \inf_{K \supset E} \mu(K)$$

   where the infimum is taken over compact K.

Also, we recall the Riesz-Markov-Kakutani (RMK) representation theorem which allows us to replace difficult Radon measures with their corresponding functionals. This is especially useful when proving the existence and up-to-scaling uniqueness of Haar measures.

Theorem 2.4. Let X be a locally compact Hausdorff space. Now the map $\mu \mapsto \mathcal{I}_\mu$, where $\mathcal{I}_\mu(f) = \int_X f d\mu$, gives a bijective correspondence between Radon measures on X and non-negative linear functionals $\mathcal{C}_c(X) \to \mathbb{R}$.

3 Haar measures

Although in the other sections we will focus on LCA groups, here we will work in the full generality of LC groups. This is because despite being not as nice, the theory of LC groups, and especially compact groups, is an important topic. For example, most Lie groups are not abelian, yet analysis on them is of great interest.

Definition 3.1. Let μ be a non-negative Radon measure on a LC group G. We call μ a *left* (resp. *right*) *Haar measure* if for all $g \in G$ and Borel-measurable $E \subset G$ we have

$$\mu(gE) = \mu(E) \text{ (resp. } \mu(Eg) = \mu(E)).$$

The theories of left and right Haar measures coincide by symmetry. Furthermore, it's not difficult to show that if μ is a left Haar measure on G, then μ' defined by

$$\mu'(U) = \mu(U^{-1}) := \mu\{g^{-1} : g \in U\}$$

is a right Haar measure on G. In the future we will assume that all Haar measures are left.

What makes a Haar measure μ especially great is the simple formula

$$\int_G f(g) d\mu(g) = \int_G f(xg) d\mu(g)$$

for all $x \in G$ and integrable f. This property follows directly from translation invariance and change of variables but it's exactly what makes most of the proofs work.

Theorem 3.2. An LC group G admits a Haar measure.

The proof for the existence of a Haar measure is Theorem 2.10. in [1]; it is not shown here since it's quite tedious and frankly, not interesting or related to harmonic analysis. The idea however is intuitive: For a compact and an open set $K, U \subset G$, the collection $\{gU\}_{g\in G}$ covers K, so by compactness there exist $g_1, \dots g_N$ so that $\{g_i U\}_{i=1,\dots,N}$ covers K. Letting $(K : U)$ be smallest such N and choosing a compact non-empty $A \subset G$ we can let $\mu_U(K) = (K : U)/(A : U)$.

Now it's easy to see that this not-really-a-measure μ_U is left translation invariant on compact sets. Also letting $U \to \{1\}$ in some sense, it's reasonable to think that μ_U starts looking like a measure. Now the rest of the proof is just pure formalizing. We will, however, prove the following important theorem:

Theorem 3.3. Haar measures on an LC group G are unique up to scaling. That is, if μ and ν are Haar measures on G, then there exists a constant $c > 0$ such that $\mu = c\nu$.

Before we get into the proof, we state the following simple lemma, the proof of which can be found from Proposition 2.19. of [1].

Lemma 3.4. For all nonempty open $U \subseteq G$ we have $\mu(U) > 0$. For all nonzero, nonnegative $f \in C_c(G)$ we have $\int_G f d\mu > 0$.

Proof of Theorem 3.3. We follow the proof given in [2]. By Theorem 2.4. it suffices to show that

$$\mathcal{I}_\mu(f)\mathcal{I}_\nu(g) - \mathcal{I}_\nu(f)\mathcal{I}_\mu(g) = 0$$

since this rewrites as a ratio. Using Lemma 2.2. we can find, for every $0 < \varepsilon < 1$ a neighborhood U_ε so that

$$|f - f(\cdot y)| < \varepsilon \text{ and } |g - g(\cdot y)| < \varepsilon$$

in all of G when $y \in U_\varepsilon$. Furthermore, by sufficiently restricting we may assume that all the U_ε all lie on a compact set K. Finally, by Urysohn's lemma and Lemma 2.4. there exists a $\psi \in C_c(G)$ supported in K, with $\int_G \psi d\mu = 1$.
Now, we have

$$\int_G f(xy)\,\psi(y)\,d\mu(y) = \mathbf{1}_{U_\varepsilon}(y)\,f(xy) = \mathbf{1}_{U_\varepsilon}(y)\,f(x) + \mathbf{1}_{U_\varepsilon}(y)\,O(\varepsilon)\,.$$

Taking $\mathcal{I}_\nu$ of both sides and using translation invariance and Fubini's theorem, we find

$$
\begin{aligned}
\int_G \int_G f(xy)\psi(y)d\mu(y)d\nu(x) = \mathcal{I}_\nu(f) + O(\varepsilon) \\
\int_G \int_G f(y)\,\psi\left(x^{-1}y\right)d\mu(y)\,d\nu(x) = \\
\int_G \int_G f(y)\,\psi\left(x^{-1}y\right)d\nu(x)\,d\mu(y) = \\
\int_G f(y)\left(\int_G \psi\left(x^{-1}y\right)d\nu(x)\right)d\mu(y) = \\
\int_G f(y)\left(\int_G \psi\left(x^{-1}\right)d\nu(x)\right)d\mu(y) = \\
\mathcal{I}_\mu(f)\cdot \int_G \psi\left(x^{-1}\right)d\nu(x) =
\end{aligned}
$$

where the $O(\varepsilon)$-term remains that way since $U_\varepsilon \subseteq K$ and K has finite measure. Similarly we have

$$\mathcal{I}_\mu(g)\cdot \int_G \psi(x^{-1})d\nu(x) = \mathcal{I}_\nu(f) + O(\varepsilon).$$

Combining these we find that

$$(\mathcal{I}_\mu(f)\mathcal{I}_\nu(g) - \mathcal{I}_\nu(f)\mathcal{I}_\mu(g))\cdot \int_G \psi(x^{-1}d\nu(x) = O(\varepsilon).$$

Finally, $\psi(\cdot^{-1})$ can be approximated below by a nonzero nonnegative function in $C_c(G)$, so $\int_G \psi(x^{-1})d\nu(x) > 0$ by Lemma 2.4. Now letting $\varepsilon \to 0$ gives us the desired equality. □

Now, let us dive into examples of Haar measures.

Example 3.5. In the discrete topology, compact sets are finite subsets. Hence if G is discrete, then it's easy to see that the counting measure gives a Haar measure on G.

Example 3.6. For any integer $n > 0$, by construction the Lebesgue measure μ retricted to Borel sets is a Radon measure. In addition, it's canonically translation invariant. Hence we find that μ is a Haar measure on $(\mathbb{R}^n, +)$.

Example 3.7. Let $G = (\mathbb{R}^+, \cdot)$, the multiplicative group on positive reals. It's easy to confirm that

$$\mu(E) = \int_E \frac{dt}{t}$$

gives a Radon measure since $1/t$ is continuous. In addition, for all (a, b) with $0 < a < b < +\infty$ we have

$$\mu(ga, gb) = \int_{ga}^{gb} \frac{dt}{t} = \log\frac{ga}{gb} = \log\frac{a}{b} = \int_a^b \frac{dt}{t} = \mu(a, b)$$

so a simple measure-theoretic argument gives that μ is a Haar measure on G.

Example 3.8. With similar reasoning, if $G = S^1$, then the measure

$$\mu(E) = \lambda(\arg(E)),$$

where λ is the Lebesgue measure on $[0, 2\pi)$, is a Haar measure on G. However, $\nu = \mu/2\pi$ is a much nicer fit for a Haar measure since $\nu(G) = 1$. In fact, we can turn any Haar measure μ on a compact group G into a probability measure just by normalizing by $\mu(G) < +\infty$!

Example 3.9. A non-abelian example! Let μ be the Lebesgue measure on $\mathbb{R}^{n\times n}$ viewed as $\mathbb{R}^{n^2}$. Now if $G = GL_n(\mathbb{R})$, then by change of variables the measure

$$\mu(E) = \int_E \frac{d\mu(X)}{|\det X|^n}$$

gives a Haar measure.

4 Pontryagin duality

From now on, all groups are assumed to be abelian.

For an LCA group G, let $\widehat{G}$ be the group of continuous homomorphisms $G \to S^1$ (characters). For all compact $K \subset G$ and $U \subset S^1$ we let $V(K, U)$ be the set of characters $\chi : G \to S^1$ with $\chi(K) \subseteq U$. We endow $\widehat{G}$ with the topology generated by the subbasis $V(K, U)$ over all K, U; the compact-open topology. Now we have the following extraordinary definition-theorem, the proof of which we will omit due to it being very tedious.

Theorem 4.1. The group $\widehat{G}$ is a topological group. Furthermore, it's LCA. We call this group the *Pontryagin dual* of G.

As one may guess from the term "duality", we also have the following important fact:

Theorem 4.2. (Pontryagin duality). The pairing $\langle -, - \rangle : G \times \widehat{G} \to S^1$ by $\langle x, \chi \rangle = \chi(x)$ gives a canonical isomorphism $e : G \to \widehat{\widehat{G}}$ by $x \mapsto \langle x, - \rangle$.

Example 4.3. When G is a finite LCA group, it must be a finite sum of $\mathbb{Z}_n$'s with discrete topologies. Since Hom interacts nicely with $\oplus$, it suffices to only consider the case $\mathbb{Z}_n$. Here we instantly see that $\widehat{\mathbb{Z}_n} \cong \mathbb{Z}_n$ as topological groups and going back, that $\widehat{G} \cong G$.

Example 4.4. If $G = (\mathbb{R}^n, +)$, then solving the given functional equation, we find that the characters are given by $f(\chi) = e^{2\pi i x \cdot \chi}$ for $x \in \mathbb{R}$. The group operation on G is isomorphic to that of $\widehat{G}$, and with a little topological work, one can verify that these groups are also homeomorphic. Hence we find that $\widehat{G} \cong (\mathbb{R}^n, +)$. This identification is usually done directly when working with the Fourier transform in analysis.

Example 4.5. Similarly, when $G = (\mathbb{R}^+, \cdot)$, we can prove that all characters are of the form $\gamma(z) = z^{ix}$ where $x \in \mathbb{R}$. Hence, as groups we have $\widehat{G} \cong (\mathbb{R}, +)$ and similarly as topological spaces. This identification is also done directly when working with the Mellin transform. A simpler way to notice this fact is by noticing that the topological groups $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic via the logarithmic map, and using the fact that the assignment $G \to \widehat{G}$ is functorial.

Example 4.6. When $G = S^1$, the characters are given by $\gamma(z) = z^n$ for $n \in \mathbb{Z}$. Hence we see that $\widehat{G} \cong \mathbb{Z}$. Furthermore, it can be shown that all compact abelian groups have a discrete Pontryagin dual. This also gives a rather interesting way to classify compact abelian groups: All compact abelian groups can be obtained simply by looking at the characters $G \to S^1$ of a (non-topological) group G and giving this the topology generated by the subbasis sets $\{\chi : G \to S^1 : \chi(K) \subset U\}$ where $K \subset G$ is finite and $U \subset S^1$ open.

5 Fourier analysis

Let us choose a fixed Haar measure for an LCA group G. Now if $f \in L^1(G)$, then we can define its Fourier transform $\hat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\chi) = \int_G \overline{\langle g, \chi \rangle} f(g) dg.$$

We denote the linear operator $f \mapsto \hat{f}$ by $\mathcal{F}$, and triangle inequality shows directly that it sends $L^1(G) \to L^0(\widehat{G})$. Furthermore, we have the following analogue of the Riemann-Lebesgue lemma:

Theorem 5.1. (Riemann-Lebesgue) $\mathcal{F}$ sends $L^1(G) \to C_0(\widehat{G})$.

This map is additionally continuous when $C_0(\widehat{G})$ is equipped with the topology induced by the sup-norm, and Stone-Weierstrass shows that the image is dense in $C_0(\widehat{G})$. The proof for Theorem 5.1 using Gelfand theory can be found from Proposition 4.13. of [1]. However, a far more intuitive proof similar, to that of the case $G = \mathbb{R}$, can be produced as follows:

Lemma 5.2. For every neighborhood $U \subseteq G$ of 1 there exists a compact set $K \subseteq \widehat{G}$, such that, if $\chi \in \widehat{G} \setminus K$, then there exists $x \in U$ with $\mathbf{Re}\chi(x) \leq 0$.

Proof. Theorem H of [3]. The proof is quite straightforward: First it's proven that if G and G' satisfy the lemma, then so does $G \times G'$. Then the lemma is proven for CA groups, then LCA groups containing an open CA, and finally for LCA groups. The lemma is easy to verify directly for, for example, the special cases $G = \mathbb{R}^n, S^1$.

Proof of Theorem 5.1. The proof for continuity is omitted but should be possible to do as with $G = \mathbb{R}$, just with harsher topological consideration. With a sufficiently nice condition of G, like first countability, the proof can be replicated completely. The proof for the vanishing is similar to the case $G = \mathbb{R}$: Let $\varepsilon > 0$ and let $U \subseteq G$ be a neighborhood of 1 such that $\|f - f(\cdot y^{-1})\|_1 < \varepsilon$ for all $y \in U$. Now we can choose K, χ and x as in Lemma 4.1. so we get

$$\overline{\chi(x)}\hat{f}(\chi) = \int_G \overline{\chi(gx)}f(g)dg = \int_G \overline{\chi(g)}f(gx^{-1})dg.$$

Now we find that

$$\hat{f}(\chi)(1 - \overline{\chi(x)}) = \int_G \overline{\chi(g)}(f(g) - f(gx^{-1})dg$$

so by triangle inequality

$$|\hat{f}| \cdot |1 - \overline{\chi(x)}| \leq \|f - f(\cdot x^{-1})\|_1 < \varepsilon.$$

Furthermore, the condition $\mathbf{Re}\chi(x) \leq 0$ forces $|1 - \overline{\chi(x)}| \geq 1$, so $|\hat{f}(\chi)| < \varepsilon$ for all $\chi \in \widehat{G} \setminus K$. But now this is exactly the definition of $C_0(\widehat{G})$! □

From the Riemann-Lebesgue lemma one can also deduce the Fourier inversion theorems.

Theorem 5.3. Let $f \in L^1(G)$ and $\hat{f} \in L^1(\widehat{G})$. There exists a Haar measure on $\widehat{G}$ such that

$$f(g) = \int_{\widehat{G}} \langle g, \chi \rangle \hat{f}(\chi)d\chi \text{ a.e..}$$

For f continuous, this holds everywhere.

Proof. Theorem 4.32. of [1]

Independently of Theorem 5.2, as in [4], one can also show that $f \in L^1(G) \cap L^2(G)$ implies $\|\hat{f}\|_2 \leq \|f\|_2$. From here we can extend $\mathcal{F}$ into an operator $L^2(G) \to L^2(\widehat{G})$ and show that Fourier inversion and Plancherel's theorem also hold in arbitrary LCA groups.

Theorem 5.4. Let $f \in L^2(G)$, and hence $\hat{f} \in L^2(\widehat{G})$. Now there exists a Haar measure on $\widehat{G}$ such that

$$f(g) = \int_{\widehat{G}} \langle g, \chi \rangle \hat{f}(\chi)d\chi \text{ a.e..}$$

For f continuous, this holds everywhere.

Proof. Theorem 3.3 of [4]

Theorem 5.5. (Plancherel) $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ is a linear isometry.

Proof. Theorem 3.4 of [4].

A cousin of Plancherel, the (almost) Hausdorff-Young inequality, also holds for LCA groups. If $f \in L^p$ with $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|\hat{f}\|_q \leq \|f\|_p.$$

In particular, $\hat{f} \in L^q(\widehat{G})$ so $\mathcal{F}$ maps $L^p(G) \to L^q(\widehat{G})$. This theorem follows directly from the Riesz-Thorin interpolation theorem, using $\|\hat{f}\|_\infty \leq \|f\|_1$ and $\|\hat{f}\|_2 \leq \|f\|_2$.

We can also define convolution as in the real case. If $f, g \in L^1(G)$, then we can let

$$(f * g)(x) = \int_G f(y)g(xy^{-1})dy = \int_G f(xy^{-1})g(y)dy = (g * f)(x)$$

where the second equality follows from translation invariance. Triangle inequality and Fubini's theorem directly give that

$$\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1.$$

Furthermore, [5] gives us Young's convolution inequality: If $f \in L^p(G), g \in L^q(G)$ where $1 \leq p, q, r \leq \infty$ with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

then

$$\|f * g\|_r \leq \|f\|_p \cdot \|g\|_q.$$

In particular, $f * g \in L^r(G)$.

Theorem 5.6. If $f, g \in L^2(G)$, then

$$\widehat{f * g} = \hat{f} \cdot \hat{g}.$$

Proof. Proposition 4.36 of [1].

In total, the compatibility of the Fourier transform and convolution is preserved even when generalizing to LCA groups.

Example 5.7. Let G is a finite LCA group, and hence with discrete topology, and $f : G \to \mathbb{C}$. The Fourier transform in this case is simply the Fourier transform for finite abelian groups

$$\hat{f}(\chi) = \sum_{g \in G} f(g)\overline{\chi(g)}.$$

Example 5.8. In the Euclidean case $G = \mathbb{R}^n$, the identification $\widehat{G} \cong \mathbb{R}^n$ gives us the classical Fourier transform

$$f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx.$$

Example 5.9. When $G = (\mathbb{R}^+, \cdot)$, then we found that the Haar measure is dt/t. Also we found that $\widehat{G}$ can be identified with $i\mathbb{R}$ by having the character $z \to z^{ix}$ correspond to ix. Now with this identification, we have

$$\hat{f}(x) = \int_0^\infty \overline{z^{i(ix)}} f(z)\frac{dz}{dt} = \int_0^\infty z^{x-1}f(z)dz.$$

But now this is simply the Mellin transform of f. Especially the case $f = e^{-x}$ is interesting where $\hat{f}$ is the Gamma function Γ from number theory.

Example 5.10. If $G = \mathbb{Z}$, then $\widehat{G} \cong S^1$ by identifying the character $n \to e^{2\pi i n}$ with $e^{2\pi i n}$. Furthermore, this can be identified with $\mathbb{R}/\mathbb{Z}$, the real numbers with period 1. In this case, the Fourier transform gives us the standard Fourier series

$$\hat{f}(z) = \sum_{n \in \mathbb{Z}} e^{-2\pi i n} f(n).$$

In conclusion, most simple and desirable properties of the standard Fourier transform, which do not run into well-definedness issues right away, also generalize for the general Fourier transform. This not only lets us unify all of the special instances of the above Fourier transforms under one theory, but it also gives great tools for many other fields which are, unfortunately, too broad to be discussed in this essay.

6 References

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