SOME TOPICS ON CARLESON MEASURES

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1. Introduction

This essay is mainly based on a serials of Brett D.Wick's lectures on Multi parameter Harmonic Analysis[1, 2, 3, 4]. In this essay we will introduce some examples of Carleson Measures both in one parameter and multi parameter situation and add some details which lectures did not mentioned carefully. And we need to claim that the essay did not cover every topics in the lectures for reason of the length of the essay, but only some parts we interested with.

As the beginning we would give a brief introduction of Carleson measures. Carleson measures were first introduced in 1960s by L.Carlseon to solve the corona problem[8] and were named after him. Carleson measures have many applications in harmonic analysis and the theory of partial differential equations. Then we start with an initial definition of Carleson measures.

Definition 1 ([7])**.** *A positive measure σ on upper half plane $\mathbb{H}$ is called a Carleson measure if there is a constant $N(\sigma)$ such that*

$$\sigma(Q) \leq N(\sigma)h$$

for all squares

$$Q = \{x_0 < x < x_0 + h, 0 < y < h\}.$$

The smallest such constant $N(\sigma)$ is the Carleson norm of σ.

In practise, we more often use its embedding property to characterise a Carleson measure, like

Theorem 1 (Theorem 3.9 of [7])**.** *Let σ be a positive measure in the upper half plane $\mathbb{H}$. Then the following are equivalent:*

(a): *σ is a Carleson measure: for some constant $N(\sigma)$,*

$$\sigma(Q) \leq N(\sigma)h$$

for all squares

$$Q = \{x_0 < x < x_0 + h, 0 < y < h\}.$$

(b): *For $0 < p < \infty$,*

$$\int |f|^p d\sigma \leq A\|f\|_{H^p}^p, \quad f \in H^p$$

(c): *For some p, $0 < p < \infty$, $f \in L^p(\sigma)$ for all $f \in H^p$.*

And we need to note that carleson measures can also be defined on $\mathbb{D}$ or $\mathbb{R}^n$, we will mention these cases in the following sections.

2. Carleson measures for $H^2(\mathbb{D})$

2.1. **some properties of $H^2(\mathbb{D})$.** We start with the introduction of Hardy space $H^2(\mathbb{D})$.

Definition 2. *For a holomorphic function f, $f \in H^2(\mathbb{D})$ if*

$$\|f\|^2_{H^2(\mathbb{D})} := \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) < \infty$$

where $m(\theta)$ is a normalized measure.

A good property of $H^2(\mathbb{D})$ is the norms of functions in it can be represented in an easier form. Recalling this, for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in L^2(\mathbb{T})$ we have

$$\begin{aligned}
\|f\|^2_{H^2(\mathbb{D})} &= \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) \\
&= \sup_{0<r<1} \int_{\mathbb{T}} |\sum_{n=0}^{\infty} a_n r^n e^{in\theta}|^2 dm(\theta) \\
&= \sup_{0<r<1} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^n r^m \int_{\mathbb{T}} e^{i(n-m)\theta} dm(\theta) \\
&= \sum_{n=0}^{\infty} |a_n|^2,
\end{aligned}$$

where the last equation is because of

$$\int_{\mathbb{T}} e^{i(n-m)\theta} dm(\theta) = \begin{cases} 1 : n = m \\ 0 : n \neq m \end{cases}$$

This property means that it is possible to study the behavior of the functions in $H^2((D))$ via their Fourier coefficients. By this property we can check the following lemma quickly.

Lemma 1. *For $0 < r < 1$ and $z \in \mathbb{D}$ let $f_r(z) = f(rz)$. Suppose that $f \in H^2(\mathbb{D})$. Then, the sequence $\{f_r\}$ is Cauchy in $L^2(\mathbb{T})$.*

Proof. Note that H^2 is a complete space. Which means that the sequence in it is Cauchy, implies that for every positive real number $b > 0$ there is a positive N such that for all $f_m, f_n \in \{f_r\}$ and $m, n > N$

$$\begin{aligned}
b &> \|f_m - f_n\|_{H^2(\mathbb{D})} \\
&\geq |\|f_m\|_{H^2(\mathbb{D})} - \|f_n\|_{H^2(\mathbb{D})}| \\
&= \left| \sum_{i=0}^{\infty} |m^i a_i|^2 - \sum_{i=0}^{\infty} |n^i a_i|^2 \right| \\
&\geq \left| \sum_{i=0}^{\infty} (m^i - n^i) a_i \right|^2 .
\end{aligned}$$

And

$$\begin{aligned}\|f_m - f_n\|_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} \left| \sum_{i=0}^{\infty} (m^i - n^i) a_i z^i \right|^2 dm(z) \\ &\leq b \int_{\mathbb{T}} \left| \sum_{i=0}^{\infty} z^i \right|^2 dm(z).\end{aligned}$$

which is convergence when $z \in \mathbb{T}$. Since b is arbitrary, then $\{f_r\}$ is Cauchy in $L^2(\mathbb{T})$. $\square$

Now since $L^2(\mathbb{T})$ is a complete space, For f_rs in $L^2(\mathbb{T})$, the function f^* denoted by $f^* := \lim_{r\to 1} f_r$ is also in $L^2(\mathbb{T})$. Then we can compute the Fourier coefficients of f^*.

$$\begin{aligned}\widehat{f^*}(n) &= \int_{\mathbb{T}} f^*(e^{i\theta} e^{-in\theta}) dm(\theta) \\ &= \lim_{r\to 1} \int_{\mathbb{T}} f_r(e^{i\theta} e^{-in\theta}) dm(\theta) \\ &= \begin{cases} a_n : n \geq 0 \\ 0 : n < 0 \end{cases}.\end{aligned}$$

Then by Parseval theorem, or say Plancherel identity, we have the following proposition.

Proposition 1. *Suppose that $f \in H^2(\mathbb{D})$ and $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ then*

$$\|f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 = \|f^*\|^2_{L^2(\mathbb{T})}.$$

And there is another norm on $H^2(\mathbb{D})$ This equivalent norm will prove useful when we we study the space of Carleson measures for $H^2(\mathbb{D})$. With it we can generate a natural family of examples of functions which generate Carleson measures.

At first, we recall the Green formula on the unit ball $\mathbb{D}$ and unit circle $\mathbb{T}$.

Lemma 2. *Green's formula:*

$$\int_{\mathbb{T}} u(\zeta) dm(\zeta) - u(0) = \int_{\mathbb{D}} \Delta u(z) \log \frac{1}{|z|} dA(z).$$

Proof. Set $v(z) = \log \frac{1}{|z|}$. By [9], Theorem 1.3.1, we have

$$\int_{\mathbb{T}} u(\nu v) - v(\nu u) d\sigma = \int_{\mathbb{D}} (u\Delta v - v\Delta u) dA.$$

where $d\sigma$ denote area measure on $\mathbb{T}$ and ν be the unit outward normal vector on $\mathbb{T}$. And then use [9], Proposition 1.3.2, which implies what we desired. $\square$

Then for a function $g \in L^1(\mathbb{T})$, the gradient of g is denoted by $\nabla g = (\partial_x g, \partial_y g)$ and when g is an analytic function, by the definition of gradient

$$|\nabla g(z)|^2 = |\partial_x g(x,y)|^2 + |\partial_y g(x,y)|^2$$

we have

$$|\nabla g(z)|^2 = |\partial g(z)|^2 = |g'(z)|^2$$

since there is no other variants. Then we introduce a Littlewood-Paley Identity.

Lemma 3. *Lemma 2.2 of* [2] *Suppose that $g \in L^1(\mathbb{T})$ and if $g(0) = \int_{\mathbb{T}} g dm$ then*

$$2\int_{\mathbb{D}} |\nabla g(z)|^2 \log \frac{1}{|z|} dA(z) = \int_{\mathbb{T}} |g - g(0)|^2 dm = \int_{\mathbb{T}} |g|^2 dm - |g(0)|^2.$$

By some rearrangement, then follows the equivalent norm

Proposition 2. *Suppose that $f \in H^2(\mathbb{D})$ then we have*

$$\|f\|^2_{H^2(\mathbb{D})} = |f(0)|^2 + 2\int_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{|z|} dA(z).$$

2.2. **Carleson embedding theorem.** And then we introduce the Carleson embedding theorem on $H^2(\mathbb{D})$:

Theorem 2. *Let μ a non-negative measure in $\mathbb{D}$. Then the following are equivalent.*

(i): *The embedding operator $\mathcal{J} : L^2(\mathbb{T}) \to L^2(\mathbb{D}, \mu)$, with $J(f)(z) = f(z)$, is bounded.*

(ii): $C(\mu)^2 := \sup_{z\in\mathbb{D}} \|\mathcal{J}\bar{k}_z\|^2_{L^2(\mu)} = \sup_{z\in\mathbb{D}} \|P_z\|_{L^1(\mu)} < \infty$, *where* $\bar{k}_z(\zeta) = \frac{(1-|z|^2)^{1/2}}{(1-\zeta\bar{z})}$, *the reproducing kernel for the Hardy space $H^2(\mathbb{D})$.*

(iii): $I_\mu = \sup\{\frac{1}{r}\mu(\mathbb{D}) \cap Q(\zeta, r) : r > 0, \zeta \in \mathbb{T}\} < \infty$, *where $Q(\zeta, r)$ is a ball measured with respect to the non-isotropic metric associated on $\mathbb{D}$.*

Moreover, the following inequalities hold

$$C(\mu) \leq \|\mathcal{J}\| \leq 4C(\mu)$$

and

$$32I(\mu) \leq C(\mu)^2 \leq 32I(\mu)$$

Since (i)$\Leftrightarrow$(iii) is [1]'s main content, we only show (ii)$\Leftrightarrow$(iii) here

Proof. Suppose (ii) is true first, then

$$C(\mu)^2 \geq \int_{\mathbb{D}} P_z(\zeta) d\mu(\zeta) = \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-z\bar{\zeta}|^2} d\mu(\zeta).$$

For $\xi \in \mathbb{T}$ and $r \in (0, 2)$, set $z = (1-\frac{r}{2})\xi$ now consider the non-isotropic ball $Q(\xi, r)$ introduced by non-isotropic metric:

$$d(z, \zeta) = |1 - z\bar{\zeta}|^{1/2}.$$

Then $z \in Q(\xi, r)$. For $z, \zeta \in Q(\xi, r)$, by triangle inequality we have

$$|1 - z\bar{\zeta}|^{1/2} \leq |1 - z\bar{\xi}|^{1/2} + |1 - \zeta\bar{\xi}|^{1/2}$$

Squaring the above inequality, get

$$|1 - z\bar{\zeta}| \leq 2(|1 - z\bar{\xi}| + |1 - \zeta\bar{\xi}|) \leq 4r$$

With this estimate, we have

$$\begin{aligned} C(\mu)^2 &\geq \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-z\bar{\zeta}|^2} d\mu(\zeta) \\ &\geq \int_{\mathbb{D}\cap Q(\xi,r)} \frac{(1-|z|^2)}{|1-z\bar{\zeta}|^2} d\mu(\zeta) \\ &\geq 16^{-1} r^{-2}(1-|z|) \int_{\mathbb{D}\cap Q(\xi,r)} d\mu(\zeta) \\ &= 16^{-1} r^{-2}(1-|z|)\mu(\mathbb{D}\cap Q(\xi, r)) \end{aligned}$$

Taking the supremum of r over $(0,2)$ in the last inequality above and get

$$32^{-1}I(\mu) \leq C(\mu)^2$$

Now suppose that (iii) is true, since the situation farer away from $\mathbb{T}$ is often more clear. We will separate the question into two cases: $|z| \leq \frac{3}{4}$ and $|z| > \frac{3}{4}$ (We note that we choose $\frac{3}{4}$ here is propose to get the estimate $32^{-1}I(\mu) \geq C(\mu)^2$, one can also choose other positive numbers less than 1).

(1).$|z| \leq \frac{3}{4}$

Recall following inequality holding for the Poisson kernel

$$P_z(\zeta) = \frac{(1-|z|^2)}{|1-z\bar{\zeta}|^2} \leq \frac{1-|z|^2}{(1-|z|)^2} = \frac{(1+|z|)^2}{(1-|z|^2)} \leq \frac{4}{(1-|z|^2)}$$

Then since $|z| \leq \frac{3}{4}$

$$\begin{aligned}\int_{\mathbb{D}} Pz(w)d\mu(w) &\leq \frac{2^6}{7}\mu(\mathbb{D}) \\ &= \frac{2^6}{7}2\mu(\mathbb{D}\cap Q(\xi,2)) \\ &\leq 20I(\mu)\end{aligned}$$

(2).$|z| > \frac{3}{4}$

Define the normalized $z' = \frac{z}{|z|}$ and

$$Q_k := \mathbb{D}\cap Q(z', 2^{k+1}(1-|z|^2)) \quad \forall k \in \mathbb{N}$$

Then for $w \in Q_{k+1} \subset Q_k$ we have

$$|1-w\bar{z'}| \geq 2^{k+1}(1-|z|^2).$$

By triangle inequality we have

$$\begin{aligned}|1-w\bar{z'}|^{1/2} &\leq |1-w\bar{z}|^{1/2} + |1-z\bar{z'}|^{1/2} \\ &= |1-w\bar{z}|^{1/2} + (1-|z|)^{1/2} \\ &\leq |1-w\bar{z}|^{1/2} + (1-|z|^2)^{1/2}\end{aligned}$$

Squaring the last inequality, get

$$|1-w\bar{z'}| \leq 2(|1-w\bar{z}| + (1-|z|^2))$$

which implies

$$\begin{aligned}|1-x\bar{z}| &\geq 2^{-1}|1-w\bar{z'}| - (1-|z|^2) \\ &\geq 2^k(1-|z|^2) - (1-|z|^2) \\ &\geq 2^{k-1}(1-|z|^2)\end{aligned}$$

When $w \in Q_{k+1} \setminus Q_k$, we have

$$\begin{aligned}\int_{\mathbb{D}} P_z(w) &= \int_{Q_1} P_z(w)d\mu(w) + \sum_{k=1}^{\infty}\int_{Q_{k+1}\setminus Q_k} P_z(w)d\mu(w) \\ &\leq \int_{Q_1} \frac{4}{(1-|z|^2)}d\mu(w) + \sum_{k=1}^{\infty}\int_{Q_{k+1}\setminus Q_k} \frac{(1-|z|^2)}{4^{k-1}(1-|z|^2)^2}d\mu(w) \\ &\leq 4\frac{\mu(Q_1)}{(1-|z|^2)} + \sum_{k=1}^{\infty}\frac{1}{4^{k-1}}\frac{\mu(Q_{k+1})}{(1-|z|^2)} \\ &\leq 16I(\mu) + 8I(\mu)\sum_{k=1}^{\infty} 2^{-k} \leq 32I(\mu)\end{aligned}$$

□

2.3. **Uchiyama's Lemma.** First we introduce a lemma which is related with embedding theorem on $H^2(\mathbb{D})$.

Lemma 4 (Uchiyama's Lemma)**.** *Let φ be a non-negative, bounded, subharmonic function. Then for any $f \in H^2(\mathbb{D})$*

$$\int_{\mathbb{D}} \tilde{\Delta}\varphi(z)|f(z)|^2 d\mu(z) \leq \|\varphi\|_\infty \|f\|_2^2$$

where $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dA(z)$, and $\tilde{\Delta} = \frac{1}{4}\Delta = \partial\bar{\partial}$

Proof. Since φ is homogeneous, we can assume without loss of generality that $\|\varphi\|_\infty = 1$, i.e. at the both side of equation, $\tilde{\Delta}\varphi$ and $\|\varphi\|_\infty$ have the same constant coefficients. By directly computation we have,

$$\tilde{\Delta}(e^{\varphi(z)}|f(z)|^2) = e^{\varphi}\tilde{\Delta}\varphi|f|^2 + e^{\varphi}|\partial\varphi f + \partial f|^2 \geq \tilde{\Delta}\varphi|f|^2$$

integrate above inequation over $\mathbb{D}$ then use Green's formula,

$$\begin{aligned}\int_{\mathbb{D}} \tilde{\Delta}\varphi|f(z)|^2 d\mu(z) &\leq \int_{\mathbb{D}} \tilde{\Delta}(e^{\varphi}|f|^2)d\mu(z) \\ &= \int_{\mathbb{T}} e^{\varphi(\zeta)}|f(\zeta)|^2 dm(\zeta) - e^{\varphi(0)}|f(0)|^2 \\ &\leq e\int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) = e\|f\|_{H^2}^2\end{aligned}$$

□

We note that if change φ to a constant function, it is easy to see that above lemma can imply the embedding $L^2_\mu(\mathbb{D}) \to H^2(\mathbb{D})$

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta)$$

3. Carleson measures in bi-disc $\mathbb{D}^2$

3.1. **Decompositions.** When researching Carleson measures, an often-used technique is to decompose the region where the Carleson measures on. There are many ways to decompose the region, e.g. Carleson tents, Carleson boxes, Carleson rectangles and so on. In practice, we can choose a most convinent decomposition to solve our problems. To decompose we usually need a lattice to satisfy a covering lemma like what follows.

Lemma 5 (Theorem 2.23 in [11])**.** *There exists a positive N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in $\mathbb{D}$ with the following properties.*

- **(1):** $\mathbb{D} = \cup_k \Delta(a_k, r)$;
- **(2):** *The set $\Delta(a_k, r/4)$ are mutually disjoint;*
- **(3):** *Each point $z \in \mathbb{D}$ belongs to at most N of the sets $\Delta(a_k, 2r)$.*

Which is a decomposition of $\mathbb{D}$, to decompose other regions is similar to this.

Now we introduce the Carleson tent. There are different ways to construct a Carleson tent. Let $Q \subset \mathbb{R}^n$ be a cube and let

$$T(Q) = \{(x, y) : x \in Q, \quad 0 \leq y \leq \ell(Q)\}$$

With this decomposition, we can give a definition of Carleson measure.

Definition 3. *A measure μ is a Carleson measure if for all cubes $Q \subset \mathbb{R}^n$, there is a constant C such that*

$$\mu(T(Q)) \leq C|Q| \tag{1}$$

And we also have another way by which we can define a Carleson measure. For simplicity, we consider case $n = 1$.

Definition 4. *Let $\Omega \subset \mathbb{R}$ be an arbitrary open set. Then a measure is Carleson if for all open sets Ω there is a C such that*

$$\mu(T(\Omega)) \leq C|\Omega|. \tag{2}$$

And then we show that in fact those two definitions are same.

Lemma 6. (1) *and*(2) *are equivalent.*

Proof. It is clear that if (2) holds then so does (1).

Suppose that (1) holds. Since Ω is a open set in $\mathbb{R}$, by some decompositions like Lemma 5 there exists disjoint open intervals $\{I_k\}$ such that $\Omega = \cup I_k$, then

$$\begin{aligned} \mu(T(\Omega)) &\leq \mu(T(\cup I_k)) \\ &\leq \sum_k \mu(T(I_k)) \\ &\leq C \sum_k |I_k| = C|\Omega| \end{aligned}$$

which implies (4), as desired. □

Also, a similar argument applies when $\Omega \subset \mathbb{R}^n$ like [5].

Now we have two possible way to study Carleson measures in one-parameter. We can either form a tent over arbitrary open sets like (4), or we can form the tent over cubes like (3). This suggests that we also have two different ways to construct Carleson tents in the multi-parameter setting.

For simplicity, we consider the case when $\Omega \subset \mathbb{R}^n$ is open. One possible candidate is

$$\mu(T(R)) \leq C|R| \text{ for } \forall R = I \times J \tag{3}$$

Here the tent over the rectangle R is given by $T(R) = T(I) \times T(J)$. Another possible candidate is

$$\mu(T(\Omega)) \leq C|\Omega| \text{ for } \forall \Omega \subset \mathbb{R}^2 \tag{4}$$

Then we consider whether these two definitions are equivalent. Again, it is immediate that (4) implies (3). But it is not clear that whether (3) could imply (4). In fact, Carleson gives a counter example about (3), which means (3) is a weaker condition.

Theorem 3 (Carleson,[8]). *There exists a measure μ such that*

$$\mu(T(R)) \leq C|R| \text{ for } \forall R = I \times J.$$

but

$$\int_{\mathbb{D}^2} |u(z,w)|^p d\mu(z,w) \leq C(p) \int_{\mathbb{T}^2} |f(t,s)|^p dtds \text{ for } \forall f \in L^p(\mathbb{T}^2)$$

3.2. **An example on bidisc $\mathbb{D}^2$.** In this section, for simplicity, we focus on the case of the bidisc $\mathbb{D}^2$.

First, recall that for an open subset $\Omega \subset \mathbb{T}^2$, we form the tent over Ω to be the union of the products of the rectangles $R = I \times J$ with $I, J \subset \mathbb{T}$ and I on the boundary of the first disc, J on the boundary of another, and $R \subset \Omega$.

$$S(\Omega) = \bigcup_{R \subset \Omega} S(I) \times S(J).$$

where $S(I)$ is the one-parameter tent over the interval I which we defined in last subsection.

Now we introduce a example of multi-parameter Carleson measure on bi-disc $\mathbb{D}^2$.

Theorem 4 (Chang[10], Fefferman[6]). *Let $f \in L^\infty(\mathbb{T}^2)$, and let $v(z,w)$ denote the multiple Poisson extension of f to the bidisc $\mathbb{D}^2$. Then for every open set $\Omega \subset \mathbb{T}^2$ we have*

$$\int\int_{S(\Omega)} |\nabla_z \nabla_w v(z,w)|^2 \log \frac{1}{|z|} \log \frac{1}{|w|} dA(z) dA(w) \lesssim |\Omega|$$

There are two different way to prove this theorem given separately by Chang[10] and Fefferman[6]. Lecture[4] introduced the approach by Fefferman, which mainly reformed $\Delta_1 \Delta_2(\varphi(u)\psi(v)) = \sum_{k,j-0}^4 \varphi^{(k)} \psi^{(j)} I_{kj}(u,v)$ (where I_{kj} is an expression involving derivatives of the functions u and v) and consider the situations of I_{kj}s separately.

Since Fefferman's method has been introduced vividly in [4], we will not mention it much here. For completeness, in this essay, we will introduce another approach by Chang[10]. Briefly speaking, he used a specialized decomposition with many useful properties to finish the proof. The original theorem of Chang is in the form like:

Theorem 5. *For a function $f \in L^\infty(\mathbb{T})$, the measure*

$$d\mu_f(z_1,z_2) = |\nabla u(z_1,z_2)|^2 \log \frac{1}{|z_1|} \log \frac{1}{|z_2|} dz_1 dz_2 d\bar{z}_1 d\bar{z}_2$$

is a Carleson measure on the bi-disc, where u is the bi-harmonic extension of f, and

$$|\nabla u(z_1,z_2)|^2 = \left(\left| \frac{\partial^2 u}{\partial z_1 \partial z_2} \right|^2 + \left| \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} \right|^2 + \left| \frac{\partial^2 u}{\partial \bar{z}_1 \partial z_2} \right|^2 + \left| \frac{\partial^2 u}{\partial \bar{z}_1 \partial \bar{z}_2} \right|^2 \right) (z_1, z_2).$$

And here is some notations. Let $\mathbb{R}_z^+ = \{z = x+iy | y > 0\}$ denote the upper half plane. For each $z = x+iy \in \mathbb{R}_z^+$, let I_z denote the interval $\{s||s-x| < y\}$ on the real line $\mathbb{R}$. For each connected open set $U \subset \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$, define $S(U)$ the region $\{z|U_z \subset U\}$. Given a function $f \in L^1(\mathbb{R}^2)$, let $u(x_1+iy_1, x_2+iy_2) = (P_{y_2} * (P_{y_1} * f))(x_1, x_2)$ denote itsnbi-harmonic extension to $\mathbb{R}_2^+ \times \mathbb{R}_2^+$, where $P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ is the Poisson kernel. Then we can reform Theorem 5 to

Theorem 6. *For a function $f \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, the measure*

$$d\mu_f(z_1, z_2) = |\nabla u(z_1, z_2)|^2 y_1 y_2 dx_1 dy_1 dx_2 dy_2$$

satisfies

$$\int\int_{S(U)} d\mu_f(z_1, z_2) \leq C|U| \text{ for all open connected sets } U \text{ in } \mathbb{R}^2. \tag{5}$$

where C is a constant only depending on the bound of f.

Then we construct a decomposition on $\mathbb{R}^2$. Fix a connected open set $U \subset \mathbb{R}^2$, let intervals I and J denote separately the projections of U to its first and second coordinate. Fix a point $x_1 \in I$, $0 \leq y_1 \leq d(x_1, I^c)$, (where I^c is the complement of I and d denotes the Euclidean distance). When $y_1 > 0$, let $\{J_{x_1,y_1,\ell}\}_\ell$ denote the collection of maximal disjoint intervals such that $\{x_1\} \times J_{x_1,0,\ell} \subset U$. For each x_1, y_2, ℓ, let $C_{x_1,y_1,\ell}$ be the centre of $J_{x_1,y_1,\ell}$ and define

$$T_{x_1,y_1}(t) = \sum_\ell \frac{|J_{x_1,y_1,\ell}|^2}{(C_{x_1,y_1,\ell} - t)^2 + |J_{x_1,y_1,\ell}|^2} \text{ for } t \in \mathbb{R}$$

For each $\alpha > 0$, let

$$E_{x_1,y_1,\alpha} = \{t \in \mathbb{R} | T_{x_1,y_1}(t) > \alpha\}.$$

Then with the following lemma we can estimate $E_{x_1,y_1,\alpha}$

Lemma 7 (Lemma 1 of [10])**.** $|E_{x_1,y_1,\alpha}| \leq (C/\alpha^{1/2}) \sum_\ell |J_{x_1,y_1,\ell}|$ *where C is a universal constant.*

And other useful properties of $E_{x_1,y_1,\alpha}$

Lemma 8 (Lemma 2 of [10])**.**

- **(a):** *If $\alpha > \beta$, then $E_{x_1,y_1,\alpha} \subset E_{x_1,y_1,\beta}$.*
- **(b):** *If $\alpha < 1/2$, then $E_{x_1,y_1,\alpha} \supset \bigcup_\ell \tilde{J}_{x_1,y_1,\ell}$, where $\tilde{J}$ means the interval with the same centre as J but twice its length.*
- **(c):** *Fix $x_1 \in I$, $0 < \alpha < 1/2$, let*

$$F_{x_1,\alpha} = \bigcup_{0 \leq y_1 \leq d(x_1, I^c)} E_{x_1,y_1,\alpha};$$

$$F_{x_1,\alpha} \setminus \bigcup_\ell \tilde{J}_{x_1,0,\ell} \subset E_{x_1,0,\alpha/8}.$$

- **(d):** *For each $0 < \alpha < 1/2$, let $U_\alpha = \bigcup_{x_1 \in I}(\{x_1\} \times F_{x_1,\alpha})$. For each $t \in \bigcup_{x_1 \in I} F_{x_1,\alpha}$, let $I_{t,\alpha}$ denote the collection of maximal intervals such that $I_{t,\alpha} \times \{t\} \subset U_\alpha$ and $I_{t,\alpha}^{x_1}$ is the interval in $I_{t,\alpha}$ which contains x_1, then*

$$t \in E_{x_1,y_1,\alpha} \Rightarrow y_1 \leq d(x_1, (I_{t,\alpha/8}^{x_1})^c).$$

- **(e):** $|U_\alpha| \leq (2 + (C/\alpha^{1/2}))|U|$ *for each $0 < \alpha < 1/2$.*

then with these we can begin to prove (5). Assume f is real-valued. Then consider

$$|\nabla u(z_1,z_2)|^2 = \left(\left|\frac{\partial^2 u}{\partial z_1 \partial z_2}\right|^2 + \left|\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2}\right|^2 + \left|\frac{\partial^2 u}{\partial \bar{z}_1 \partial z_2}\right|^2 + \left|\frac{\partial^2 u}{\partial \bar{z}_1 \partial \bar{z}_2}\right|^2\right)(z_1,z_2).$$

take $z = x + iy$, we get

$$|\nabla u(x_1+iy_1, x_2+iy_2)|^2 = \left(\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_1 \partial y_2}\right|^2 + \left|\frac{\partial^2 u}{\partial y_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial y_1 \partial y_2}\right|^2\right)(z_1,z_2).$$

Then we only need to consider the term $\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2$. Proof on other three terms are same.

For the fixed open, connected set U, let $J^{x_2}_{x_1,y_1,\ell}$ denote the interval $J_{x_1,y_1,\ell}$ which contains x_2. Then integrate by parts we get

$$\begin{aligned}
&\int\int_{S(U)} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_i+iy_1, x_2+iy_2)\right|^2 y_1 y_2 dx_1 dy_1 dx_2 dy_2 \\
\leq\ & \int_I \int_0^{d(x_1,I^c)} \left(\int_{\bigcup J_{x_1,y_1,\ell}} \left(\int_0^{d(x_2,\left(J^{x^2}_{x_1,y_1,\ell}\right)^c)} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_i+iy_1, x_2+iy_2)\right|^2 y_2 dy_2\right) dx_2\right) y_1 dy_1 dx_1
\end{aligned}$$

Fix x_1, y_1, consider the middle term

$$A_{x_1,y_1} := \int_{\bigcup J_{x_1,y_1,\ell}} \int_0^{d(x_2, J^{x^2}_{x_1,y_1,\ell})^c} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 y_2 dy_2 dx_2$$

Choose some $0 < \alpha_0 < 1/2$ and write $f = f^{(1)}_{x_1,y_1} + f^{(2)}_{x_1,y_1}$, where

$$f^{(1)}_{x_1,y_1} = f\chi_{\mathbb{R}\times E_{x_1,y_1,\alpha_0}}, \quad f^{(2)}_{x_1,y_1} = f - f^{(1)}_{x_1,y_1}.$$

Then we consider $f^{(1)}_{x_1,y_1}$, let $A^{(1)}_{x_1,y_1}, A^{(2)}_{x_1,y_1}$ be the same form as A_{x_1,y_1} with respect to the bi-harmonic extensions $u^{(1)}_{x_1,y_1}$, $u^{(2)}_{x_1,y_1}$ of $f^{(1)}_{x_1,y_1}$ and $f^{(2)}_{x_1,y_1}$ separately. Then

$$\begin{aligned}
A^{(1)}_{x_1,y_1} &\leq \int_{\mathbb{R}} \int_0^\infty |\frac{\partial^2 u^{(1)}_{x_1,y_1}}{\partial x_1 \partial x_2}(x_i+iy_1, x_2+iy_2)|^2 y_2 dy_2 dx_2 \\
&\lesssim \int_{\mathbb{R}} \left|\frac{\partial^2 u^{(1)}_{x_1,y_1}}{\partial x_1}(x_i+iy_1, t)\right|^2 dt \\
&\lesssim \int_{E_{x_1,y_1,\alpha_0}} \left|\frac{\partial u}{\partial x_1}(x_1+iy_1, t)\right|^2 dt.
\end{aligned}$$

Here $u^{(1)}_{x_1,y_1}$, u respectively denotes the harmonic extension in t to $\mathbb{R}^2_+$ of the functions $f^{(1)}_{x_1,y_1}(x_1+iy_1,t)$ and $f(x_1+iy_1,t)$. Thus by inequality above, change the order of integrate, Lemma 8 (d) we have

$$\begin{aligned}
&\int_I \int_0^{d(x_1,I^c)} A^{(1)}_{x_1,y_1} y_1 dy_1 dx_1 \\
\lesssim\ & \int_I \int_0^{d(x_1,I^c)} \int_{E_{x_1,y_1,\alpha_0}} \left| \frac{\partial u}{\partial x_1}(x_1+iy_1,t) \right|^2 dt y_1 dy_1 dx_1 \\
\lesssim\ & \int_I \int_{\cup_{0\le y\le d(x_1,I^c)} E_{x_1,y_1,\alpha_0}} \left(\int_{\{y_1 | t\in E_{x_1,y_1,\alpha_0}\}} \left| \frac{\partial u}{\partial x_1}(x_1+iy_1,t) \right|^2 y_1 dy_1 \right) dt dx_1 \\
\lesssim\ & \int \int_{U_{\alpha_0}} \left(\int_0^{d(x_1,(I^{x_1}_{t,\alpha_0/8})^c)} \left| \frac{\partial u}{\partial x_1}(x_1+iy_1,t) \right|^2 y_1 dy_1 \right) dt dx_1 \\
\le\ & \int \int_{U_{\alpha_0/8}} \left(\int_0^{d(x_1,(I^{x_1}_{t,\alpha_0/8})^c)} \left| \frac{\partial u}{\partial x_1}(x_1+iy_1,t) \right|^2 y_1 dy_1 \right) dt dx_1 \\
=\ & \int_{J_{\alpha_0/8}} \int_{I_{t,\alpha_0/8}} \left(\int_0^{d(x_1,(I^{x_1}_{t,\alpha_0/8})^c)} \left| \frac{\partial u}{\partial x_1}(x_1+iy_1,t) \right|^2 y_1 dy_1 \right) dt dx_1 \\
\lesssim\ & \|f\|_\infty^2 \int_{J_{\alpha_0/8}} |I_{t,\alpha_0/8}| dx = |U_{\alpha_0/8}| \|f\|^2_\infty
\end{aligned}$$

where $J_{\alpha_0/8}$ is the set which is the projection of $J_{\alpha_0/8}$ mapping on t(the second coordinate).

And then we estimate the second term $A^{(2)}_{x_1,y_1}$. Again, let $u^{(2)}_{x_1,y_1}, u$ denote the harmonic extension in the t variable of $f^{(2)}_{x_1,y_1}(x_1+iy_1,t)$, $f(x_1+iy_1,t)$ respectively. Then by directly gradient calculate we get

$$\begin{aligned}
\left| \frac{\partial^2 u^{(2)}_{x_1,y_1}}{\partial x_1 \partial x_2}(x_1+iy_1, x_2+iy_2) \right| &\le \int_{\mathbb{R}} \frac{|(\partial u^{(2)}_{x_1,y_1}/\partial x_1)(x_1+iy_1,t)|}{(x_2-t)^2+y_2^2} dt \\
&\le \int_{(E_{x_1,y_1,\alpha_0})^c} \frac{|(\partial u/\partial x_1)(x_1+iy_1,t)|}{(x_2-t)^2+y_2^2} dt.
\end{aligned}$$

By Lemma 8 (b), $t \in (E_{x_1,y_1,\alpha_0})^c$ implies $t \in \left(\bigcup_\ell \widetilde{J}_{x_1,y_1,\alpha} \right)^c$. Which implies that for $x_2 \in J_{x_1,y_1,\ell}$ and $y_2 \le d(x_2, (J^{x_2}_{x_1,y_1,\ell})^c)$, one has $(x_2-t)^2 + y_2^2 \gtrsim (C^{x_2}_{x_1,y_1,\ell} - t)^2 + |J^{x_2}_{x_1,y_1,\ell}|^2$.(recall that $C_{x_1,y_1,\ell}$ is the centre of $J_{x_1,y_1,\ell}$). By this inequality and directly estimate we get

$$\begin{aligned}
&\int_0^{d(x_2,(J^{x_2}_{x_1,y_1,\ell})^c)} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1+iy_1, x_2+iy_2) \right|^2 y_2 dy_2 \\
\lesssim\ & \left(\int_{E^c_{x_1,y_1,\alpha_0}} \frac{\left| \partial u/\partial x_1(x_1+iy_1,t) |J^{x_2}_{x_1,y_1,\ell}| \right|}{(C^{x_2}_{x_1,y_1,\ell}-t)^2 + |J^{x_2}_{x_1,y_1,\ell}|^2} dt \right)^2 \\
\lesssim\ & \int_{E^c_{x_1,y_1,\alpha_0}} \left| \frac{\partial u}{\partial x_1}(x_1+iy_1,t) \right|^2 \frac{|J^{x_2}_{x_1,y_1,\ell}|}{(C^{x_2}_{x_1,y_1,\ell}-t)^2 + |J^{x_2}_{x_1,y_1,\ell}|^2} dt
\end{aligned}$$

Recalling that $T_{x_1,y_1}(t) = \sum_\ell \frac{|J_{x_1,y_1,\ell}|^2}{(C_{x_1,y_1,\ell}-t)^2+|J_{x_1,y_1,\ell}|^2}$ for $t \in \mathbb{R}$, thus

$$A^{(2)}_{x_1,y_1} \lesssim \int_{E^c_{x_1,y_1,\alpha_0}} \left|\frac{\partial u}{\partial x_1}(x_1+iy_1,t)\right|^2 T_{x_1,y_1}(t)dt$$

By 8 (a), choose a decreasing sequence α_n which tends to zero, by the definition of E_{x_1,y_1,α_n}s one has

$$\begin{aligned} A^{(2)}_{x_1,y_1} &\lesssim \sum_{n=1}^{\infty} \int_{E^c_{x_1,y_1,\alpha_n \setminus x_1,y_1,\alpha_{n-1}}} \left|\frac{\partial u}{\partial x_1}(x_1+iy_1,t)\right|^2 T_{x_1,y_1}(t)dt \\ &\leq \sum_{n=1}^{\infty} \alpha_{n-1} \int_{E_{x_1,y_1,\alpha_n}} \left|\frac{\partial u}{\partial x_1}(x_1+iy_1,t)\right|^2 dt \end{aligned}$$

Hence

$$\int_I \int_0^{d(x_1,I^c)} A^{(2)}_{x_1,y_1} y_1 dy_1 dx_1 \leq \sum_{n=1}^{\infty} B_n$$

where

$$B_n = \alpha_{n-1} \int_I \int_0^{d(x_1,I^c)} \int_{E_{x_1,y_1,\alpha_n}} \left|\frac{\partial u}{\partial x_1}(x_1+iy_1,t)\right|^2 dt y_1 dy_1 dx_1$$

And one can estimate B_n by the same way used to estimate $A^{(1)}_{x_1,y_1}$. Then get

$$B_n \leq \alpha_{n-1}|U_{\alpha_n/8}|\|f\|^2_\infty.$$

With these estimates above and 8 (e), get

$$\begin{aligned} \int\int_{S(U)} |\frac{\partial^2 u}{\partial x_1 \partial x_2}|^2 y_1 y_2 dy_1 dy_2 dx_1 dx_2 &\lesssim \left(|U_{\alpha_0/8}| + \sum_{n=1}^{\infty} \alpha_{n-1}|U_{\alpha_n/8}|\right)\|f\|^2_\infty \\ &\lesssim C|U| \end{aligned}$$

And sodo this to other three terms, which implies what we desired.

4. Conclusion

In this essay, we introduced Carleson measures on $H^2(\mathbb{D})$ and Carleson measures on bi-disc $\mathbb{D}^2$ respectively as an example of single-parameter Carleson measures and multi-parameter Carleson measures. By this examples we find that when in the case of multi-parameters, many properties and techniques is still valid or has a multi-parameter edition. But even in the space simple as $\mathbb{D}^2$, calculate is much more complicated than one-parameter cases. Also, choose a proper decomposition will make the proof easier.

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