University of Helsinki

Master's thesis

Singular Integrals and Beltrami Type Operators in the Plane and Beyond

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Contents

1 Introduction

This thesis is intended to complement and extend the authors candidate thesis, "The Hilbert transform" [Kos10]. This candidate thesis is not necessarily a prerequisite, but the reader is expected to be familiar with the basic theory of singular integral operators, as well as with complex and functional analysis up to graduate level. The appendix at the end contains notes on the notation and definitions used throughout the thesis.

We recall that the Hilbert transform of a sufficiently well-behaved function f on $\mathbb{R}$ was defined by the principal value integral

$$\mathscr{H}f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(x-t)}{t}dt.$$

In [Kos10] we extended the domain of the Hilbert transform to the L^p-spaces for $1 \leq p < \infty$, and in fact showed that this operator is bounded in the sense that $||\mathscr{H}f||_p \leq ||\mathscr{H}||_p||f||_p$ for some constants $||\mathscr{H}||_p$ whenever $1 < p < \infty$. The first question we will address in this thesis is why the L^p-boundedness of the Hilbert transform is of such fundamental importance. We will do this by considering various other singular integral operators and relating their L^p-norms to the norms of $\mathscr{H}$ by exact and approximate results. These include the complex Riesz transform and its square, the Beurling transform. For completeness we mention that the exact norms of the Hilbert transform on the L^p-spaces are given by

$$||\mathscr{H}||_p = \begin{cases} \tan(\frac{\pi}{2p}), & 1 < p < 2 \\ \cot(\frac{\pi}{2p}), & 2 \leq p < \infty \end{cases}, \tag{1.1}$$

which is a result due to Pichorides [Pic72].

Of course, a reader might wonder what kind of applications the study of mapping properties of singular integral operators might have, and it is why the remaining sections will focus on showing some of the ways to apply this theory. One of the classical applications of singular integrals may be found in the field of elliptic partial differential equations, whose solvability often crucially depends on estimates for the operator norms of related operators. We will mostly focus on the Beltrami equation, or more appropriately, the L^p-theory of the operators arising from it and its generalizations. We will introduce and invert the complex and real-linear versions of these operators under a uniform ellipticity assumption. The proofs of invertibility will be using functional analytic methods based on, for example, the theory of the Fredholm index. A major role in these proofs will be played by a compactness result on commutators of Calderón-Zygmund integral operators with VMO-functions, a result which is usually attributed to the papers of [CRW76] and [Uch78]. After we are done inverting the basic cases, these results will be further generalized to the setting where the range of our functions is higher-dimensional.

The last of our sections, Section 5, also extends many of the previous results to the setting of Clifford analysis. Unlike in the previous sections, we do not work on the complex plane but instead on the four-dimensional algebra of quaternions. The main focus of this section will be the generalization of the Cauchy and Beurling transforms to the quaternionic setting and the L^p-theory of these generalized operators. As a result of these efforts we are able to solve a quaternionic version of the basic inhomogenous Beltrami equation in the familiar L^p-space setting.

1.1 Acknowledgements

Although we claim that most of the results are original, a strong source of guidance and inspiration has been the book of Astala, Iwaniec and Martin, [AIM09]. It has certainly been a pleasure extending and recreating some of the beautiful proofs of this book.

As the author, I would like to give special thanks to my advisor, professor Tadeusz Iwaniec, who certainly picked out fertile ground for the growth of this thesis. It is said that the greatest of advisors are able to direct their students to topics that they can naturally find interest in and expand upon, and I have been fortunate to be a part of this process. I would also like to express my deepest of gratitudes to the support that I have received from him outside of mathematics.

During the finishing stages of the thesis I also received lots of sharp and perceptive comments from the second reader, István Prause, whose contribution is also greatly appreciated.

In closing I would like to thank my colleague Jarmo Jääskeläinen for sharing his knowledge of the Beltrami equation during the discussions we had, and giving useful advice on research in general. But for now, let us proceed with the mathematics.

2 Multiplier operators

Recall that for L^2-functions f the Hilbert transform satisfies a Fourier-multiplier formula, i.e.

$$\widehat{\mathscr{H}f}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

This identity combined with Parseval's theorem shows that $\mathscr{H}$ is an isometry on $L^2(\mathbb{R},\mathbb{C})$ and hence also a bounded operator on this space. In general, if we are given a function m in $L^\infty(\mathbb{R}^n)$, we may define the associated **multiplier operator** P_m on $L^2(\mathbb{R}^n,\mathbb{C})$ by the formula

$$\widehat{P_m f}(\xi) = m(\xi)\hat{f}(\xi).$$

Such an operator is immediately seen to be linear, and the essential boundedness of m will imply that P_m is a bounded operator on L^2. Often, however, the real question one wants to ask here is if P_m can be extended to a bounded operator from L^p to itself for $1 < p < \infty$. In general the question of characterizing all multipliers m for which the associated operator is bounded on $L^p(\mathbb{R}^n, \mathbb{C})$ remains unknown, although there are some conditions that have been proven to be sufficient, such as the Mikhlin multiplier theorem ([Ste70], page 96). The following immediate result will be of use to us later.

Theorem 2.1. *Let $m_1, m_2, \cdots \in L^\infty$ be a uniformly bounded sequence of functions that converge to the function m pointwise almost everywhere. Then for $1 < p < \infty$,*

$$||P_m||_p \leq \liminf_{n\to\infty} ||P_{m_n}||_p. \tag{2.1}$$

Proof. Fix first $p \geq 2$, and suppose without loss of generality that the right hand side of (2.1) is finite. By passing to a subsequence, we may assume that $||P_{m_n}||_p$ converges to some finite value as $n \to \infty$. Let now $f \in C_0^\infty$ be given. We claim that

$$||P_m f - P_{m_n} f||_p \to 0$$

as $n \to \infty$. By the Hausdorff-Young inequality

$$||P_m f - P_{m_n} f||_p \leq \left|\left|(m - m_n)\hat{f}\right|\right|_q.$$

Now the uniform boundedness of the m_n and dominated convergence show that this converges to zero as $n \to \infty$. We conclude that

$$||P_m f||_p = \lim_{n\to\infty} ||P_{m_n} f||_p \leq ||f||_p \lim_{n\to\infty} ||P_{m_n}||_p,$$

which gives (2.1). The case $p < 2$ is handled by a standard duality argument. □

Some common singular integrals given by L^2-multipliers include the Riesz transforms defined as

$$R_j f(x) = c_n \text{p.v.} \int_{\mathbb{R}^n} \frac{(x_j - y_j) f(y)}{|x - y|^{n+1}} dy,$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$ is a normalizing constant. The Riesz transforms may be thought of as n-dimensional analogues of the Hilbert transform, and each R_j is known to be given by the multiplier $-i\xi_j/|\xi|$.

For a large part of the thesis the setting will be the complex plane, mostly because the planar theory is more well-behaved than its n-dimensional analogues, but also for the purpose of easy visualization (see for example Figures

1 and 2). A singular integral of special interest to us will be the complex Riesz transform defined by

$$\mathcal{R}f = R_2 + iR_1, \tag{2.2}$$

where R_1 and R_2 are the two-dimensional Riesz transforms. We may calculate the multiplier of $\mathcal{R}$ as

$$\widehat{\mathcal{R}f}(\xi) = \left(-i\frac{\xi_2}{|\xi|} + \frac{\xi_1}{|\xi|}\right)\hat{f}(\xi) = \frac{\overline{\xi}}{|\xi|}\hat{f}(\xi).$$

If we apply $\mathcal{R}$ twice we get an operator with multiplier $\overline{\xi}^2/|\xi|^2$. This operator, denoted by $\mathcal{S}$, is called the Beurling transform and it may also be given by the principal value integral

$$\mathcal{S}f(z) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}} \frac{f(\omega)}{(z-\omega)^2}d\omega.$$

The later sections will demonstrate the fundamental importance of the Beurling transform in applications to partial differential equations, but for now we set the applications aside. Instead we will focus on the question of L^p-boundedness of our operators. Sometimes just boundedness isn't enough though; one often also needs the precise values of the L^p-operator norms. For example the norms of the Riesz transforms are known to equal the norm of the Hilbert transform, proven for example in [IM96], but the norms of the complex Riesz transform still seem to be unknown, as is the case for the Beurling transform. A conjecture of Tadeusz Iwaniec states that these norms are given by

$$||\mathcal{S}||_p = \begin{cases} \frac{1}{p-1}, & 1 < p < 2 \\ p-1, & 2 \le p < \infty \end{cases}.$$

2.1 The operators T_k

Still working in the plane, we now explore a certain sequence of operators whose multipliers converge pointwise a.e. to the multiplier $\overline{\xi}/|\xi|$ of the complex Riesz transform. The main aim here is to apply Theorem 2.1 to obtain bounds on the p-norms of $\mathcal{R}$. Let us first define for each angle θ the *directional Hilbert transform* by

$$\mathscr{H}_\theta f(z) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(z - te^{i\theta})\frac{dt}{t}.$$

The first of our operators is defined by $T_1 = i\mathscr{H}_{\pi/2}$. If we use the notation $g_\theta(z) = g(e^{i\theta}z)$ for any function g, then we may write

$$\mathscr{H}_{\theta+\pi/2}f = -i\left(T_1(f_\theta)\right)_{-\theta},$$

which in fact shows that $||\mathscr{H}_\theta||_p = ||T_1||_p$ for any θ and $1 < p < \infty$. The multiplier of T_1 is then given by the computation

$$\begin{aligned}\widehat{T_1 f}(\xi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_1 f(x,y) e^{i(\xi_1 x + \xi_2 y)} dxdy \\ &= \int_{-\infty}^{\infty} e^{i\xi_1 x}\left(\int_{-\infty}^{\infty} T_1 f(x,y) e^{i\xi_2 y} dy\right) dx \\ &= \int_{-\infty}^{\infty} e^{i\xi_1 x}\operatorname{sgn}(\xi_2)\left(\int_{-\infty}^{\infty} f(x,y) e^{i\xi_2 y} dy\right) dx = \operatorname{sgn}(\xi_2)\hat{f}(\xi),\end{aligned}$$

which leads us to define the multiplier $m_1(\xi) = \operatorname{sgn}(\xi_2)$. As the Fourier transform obeys the formula

$$\widehat{(f_\theta)} = (\hat{f})_\theta$$

for any angle θ, we have as a corollary of the above calculation that

$$\widehat{\mathscr{H}_\theta f}(\xi) = -i m_1(i e^{-i\theta}\xi)\hat{f}. \tag{2.3}$$

Turning now to the question of L^p-boundedness, it naturally happens that the p-norms of T_1 and $\mathscr{H}$ agree, which is proven in [IM96] for example. This also shows that the p-norms of the directional Hilbert transforms and $\mathscr{H}$ are the same, which will turn out to be of value soon.

Let now k be a given positive integer. We denote by $\epsilon = e^{\pi i/k}$ the $2k$:th root of unity. Let us also define the angles $\theta_j = \pi(j+1)/k - \pi/2$ for $j = 0, 1, \ldots, k-1$. Then we define the operator T_k by

$$T_k f = i C_k \sum_{j=0}^{k-1} \epsilon^{-j}\mathscr{H}_{\theta_j} f, \tag{2.4}$$

where C_k is a constant given by $C_k = (1-\epsilon^{-1})/2$, so that $|C_k| = \sin(\pi/2k)$. The multiplier m_k of T_k is given by (2.3)

$$m_k(\xi) = i C_k \sum_{j=0}^{k-1} \epsilon^{-j}(-i) m_1(i e^{-i\theta_j}\xi) = C_k \sum_{j=0}^{k-1} \epsilon^{-j} m_1(\epsilon^{k-j-1}\xi),$$

but there is a simpler form available. Note that the function $\epsilon^{-j} m_1(\epsilon^{k-j-1}\xi)$ attains the value ϵ^{-j} in the half-plane where $\operatorname{Im}(\epsilon^{k-j-1}\xi) > 0$ and $-\epsilon^{-j}$ in the opposite half-plane. The boundary of these half-planes is the line spanned by ϵ^{j+1-k}. Collecting these lines for each j, we obtain k lines passing through the origin that divide the complex plane into $2k$ sectors of equal angles, which we denote by $S_0, S_1, \ldots, S_{2k-1}$ starting from the positive x-axis and counting in the positive direction. By a geometric consideration (see Figure

1 below) one verifies that the value of m_k in the sector S_j is the constant given by

$$m_k(\xi) = C_k \sum_{l=j}^{k-1} \epsilon^{-l} - C_k \sum_{l=0}^{j-1} \epsilon^{-l} = C_k \left(\epsilon^{-j} \frac{\epsilon^{j-k} - 1}{\epsilon^{-1} - 1} - \frac{\epsilon^{-j} - 1}{\epsilon^{-1} - 1} \right)$$

$$= C_k \frac{\epsilon^{-k} + 1 - 2\epsilon^{-j}}{\epsilon^{-1} - 1} = C_k \frac{2\epsilon^{-j}}{1 - \epsilon^{-1}} = \epsilon^{-j}$$

for all $\xi \in S_j$.

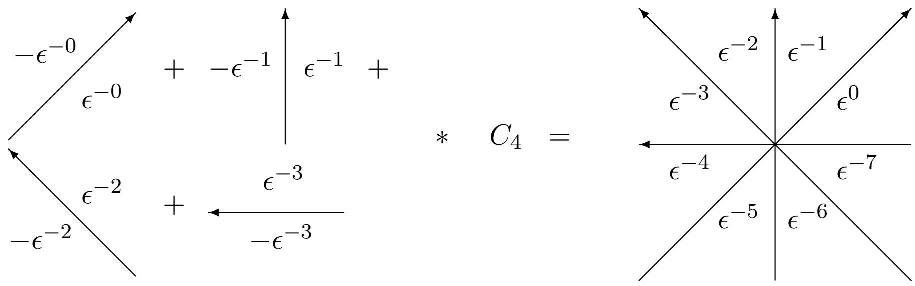

Figure 1: Illustration of the case $k = 4$.

Thus if we express ξ in polar coordinates, $\xi = re^{i\theta}$ with $0 < \theta < 2\pi$, then the value of m_k at ξ is $e^{-\pi i l/k}$, where l is the unique integer such that $l/k \leq \theta < (l+1)/k$. This shows that the functions $m_k(\xi)$ indeed converge to $\overline{\xi}/|\xi|$ pointwise almost everywhere as $k \to \infty$.

2.2 Upper bounds

We now seek upper estimates for the p-norms of the operators T_k. We denote the p-norm of an operator T restricted to real-valued functions by $||T||_p^{\mathbb{R}}$. We first concern ourselves with the operator T_2, which may be written as

$$T_2 = \frac{1-i}{2} \left(\mathscr{H}_0 - i\mathscr{H}_{\pi/2} \right).$$

Let $p \geq 2$, and a real-valued function f in L^p be given. The following estimate is imminent

$$||T_2 f||_p = \frac{1}{\sqrt{2}} ||\mathscr{H}_0 f - i\mathscr{H}_{\pi/2} f||_p = \frac{1}{\sqrt{2}} ||(\mathscr{H}_0 f)^2 + (\mathscr{H}_{\pi/2} f)^2||_{p/2}^{1/2}$$

$$\leq \frac{1}{\sqrt{2}} \left(||(\mathscr{H}_0 f)^2||_{p/2} + ||(\mathscr{H}_{\pi/2} f)^2||_{p/2} \right)^{1/2}$$

$$= \frac{1}{\sqrt{2}} \left(||\mathscr{H}_0 f||_p^2 + ||\mathscr{H}_{\pi/2} f||_p^2 \right)^{1/2} \leq ||\mathscr{H}||_p ||f||_p.$$

This enables us to recover the estimate $||T_2||_p^{\mathbb{R}} \leq ||\mathscr{H}||_p$ (extend to $1 < p < 2$ by duality). In fact, we later show that this bound is optimal. But let us first concern ourselves with how this may be used to recover estimates for the other operators T_k.

Let $k \in \mathbb{Z}_+$ be even, and ϵ be the $2k$:th root of unity, so $\epsilon^{k/2} = i$. Note that for even k we may write equation (2.4) in the form

$$T_k = iC_k \sum_{j=0}^{k/2-1} \epsilon^{-j}(\mathscr{H}_{\theta_j} - i\mathscr{H}_{\theta_j+\pi/2}).$$

Now each of the operators $\mathscr{H}_{\theta_j} - i\mathscr{H}_{\theta_j+\pi/2}$ may be viewed as a rotation of the operator $\mathscr{H}_0 - i\mathscr{H}_{\pi/2}$, and thus we find the estimate

$$||T_k||_p^{\mathbb{R}} \leq |C_k| \sum_{j=0}^{k/2-1} ||\mathscr{H}_0 - i\mathscr{H}_{\pi/2}||_p^{\mathbb{R}} \leq \sin\left(\frac{\pi}{2k}\right)\frac{k}{2}\sqrt{2}||\mathscr{H}||_p.$$

As $k \to \infty$, the utmost right hand side converges to $\pi/(2\sqrt{2})||\mathscr{H}||_p$, from which we find the following estimate by Theorem 2.1

$$||\mathcal{R}||_p^{\mathbb{R}} \leq \frac{\pi}{2\sqrt{2}}||\mathscr{H}||_p. \tag{2.5}$$

Another more straightforward estimate is

$$||T_k||_p \leq |C_k| \sum_{j=0}^{k-1} ||\mathscr{H}_{\theta_j}||_p \leq \sin\left(\frac{\pi}{2k}\right) k||\mathscr{H}||_p \leq \frac{\pi}{2}||\mathscr{H}||_p,$$

from which we get

$$||\mathcal{R}||_p \leq \frac{\pi}{2}||\mathscr{H}||_p. \tag{2.6}$$

These bounds for $||\mathcal{R}||_p$ are of course not optimal, as $\mathcal{R}$ is an isometry on L^2 for example. But especially (2.5) is close, since we will soon find the lower bound $||\mathcal{R}||_p^{\mathbb{R}} \geq ||\mathscr{H}||_p$, and $\pi/(2\sqrt{2}) \approx 1.11072$. Note that (2.6) also implies the boundedness of the Beurling transform, as in fact we get the estimate

$$||\mathcal{S}||_p \leq \frac{\pi^2}{4}||\mathscr{H}||_p^2. \tag{2.7}$$

Remark 2.1. Note that the bounds for the p-norms of the operator $\mathcal{S}$ in (2.7) are also far from optimal. Plugging the exact values of the norms $||\mathscr{H}||_p$ as given by (1.1) shows that the right hand side of (2.7) is asymptotically equivalent to p^2 for large p, while the conjectured bounds (and the best known bounds, [BJ08]) are linear in p.

2.3 Lower bounds

We now try to find some lower bounds in a more general setting. Let T_Ω be any singular integral given by convolution against the kernel

$$K_\Omega(z) = \frac{\Omega(z/|z|)}{|z|^2},$$

where Ω is some odd integrable function on the unit circle. The main aim of this subsection is to provide a certain nontrivial lower bound for the operator norms $||T_\Omega||_p$. Note that choosing $\Omega(z) = \frac{i}{2\pi}\overline{z}$ gives the complex Riesz transform, which we recall was defined by 2.2. The well-known method of rotations gives us the integral representation

$$T_\Omega(f)(z) = -\frac{\pi i}{2}\int_0^{2\pi} \Omega(e^{i\theta})\mathscr{H}_\theta(f)(z)d\theta. \tag{2.8}$$

To obtain our lower bound we would like to set some of the $\mathscr{H}_\theta f$ to be equal. An educated guess leads us to investigate functions f of the form $f(x,y) = g(x+y)$, where g is smooth and compactly supported on the real line. For every $-\pi/4 < \theta < 3\pi/4$ we have that $\cos(\theta) + \sin(\theta) > 0$ and thus we find the identity

$$\mathscr{H}_\theta f(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x+y-(\cos(\theta)+\sin(\theta))t)\frac{dt}{t} = \mathscr{H}g(x+y).$$

We note that the right hand side indeed does not depend on θ. The problem here is however that such a function f is almost never integrable, which prompts us to use a limiting argument. The following theorem will be our aim.

Theorem 2.2. *Let Ω be an odd integrable function on the unit circle. Then*

$$||T_\Omega||_p \geq \pi \left| \int_C \Omega(e^{i\theta})d\theta \right| ||\mathscr{H}||_p,$$

where $C \subset \partial B(0,1)$ is any semicircle.

Proof. Let g be as before, and choose $k > 0$ such that $\operatorname{supp}(g) \subset (-k,k)$. Also define for large enough $R > 0$ the strip

$$S_R = \{(x,y) \in \mathbb{R}^2 : |x-y| < R\}$$

and the associated rectangle contained in S_R

$$N_R = \{(x,y) \in \mathbb{R}^2 : |x-y| < R - R^{\frac{3}{4}}, |x+y| < \frac{1}{2}R^{\frac{1}{4}} - k\}.$$

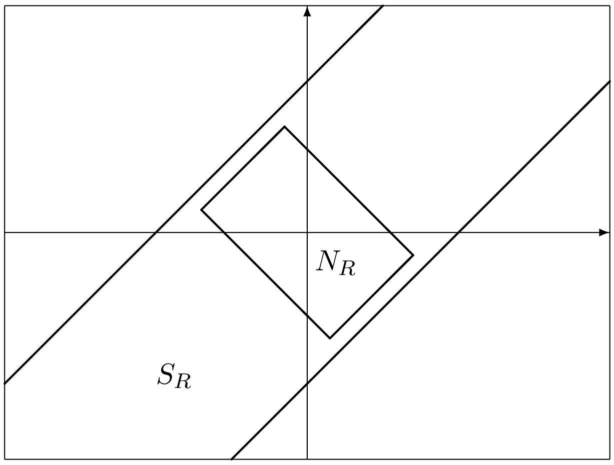

Figure 2: The strip S_R and the rectangle N_R.

To give some idea why we have chosen sets of this type, let $h(x,y) = \tilde{h}(x+y)$ be a function in $\mathbb{R}^2$, where $\tilde{h}$ is integrable on $\mathbb{R}$. A change of variables shows that

$$\int_{S_R} |h(x,y)|^p dxdy = \int_{-R}^{R}\int_{-\infty}^{\infty} \frac{1}{2}|\tilde{h}(t)|^p dtds = R||\tilde{h}||_p^p,$$

and

$$\int_{N_R} |h|^p dxdy = \int_{-R+R^{3/4}}^{R-R^{3/4}} \int_{-\frac{1}{2}R^{1/4}+k}^{\frac{1}{2}R^{1/4}-k} \frac{1}{2}|\tilde{h}(t)|^p dtds = (R - R^{\frac{3}{4}})||\chi_R \tilde{h}||_p^p,$$

where χ_R is the characteristic function of the interval $(-\frac{1}{2}R^{\frac{1}{4}}+k, \frac{1}{2}R^{\frac{1}{4}}-k)$. Nevertheless, both of these are asymptotically the same, i.e.

$$\lim_{R\to\infty} \frac{||\chi_{S_R}(x,y)h(x,y)||_p^p}{R} = \lim_{R\to\infty} \frac{||\chi_{N_R}(x,y)h(x,y)||_p^p}{R} = ||\tilde{h}||_p^p \qquad (2.9)$$

Denote as before $f(x,y) = g(x+y)$. Let us make the following claim.

Claim. For every θ in the interval $-\pi/4 < \theta < 3\pi/4$ if R is chosen large enough, specifically $R^{1/2} > |\cos\theta + \sin\theta|^{-1}$, the directional Hilbert transform $\mathscr{H}_\theta$ of $\chi_{S_R} f$ agrees with $\mathscr{H} g(x+y)$ for $(x,y) \in N_R$.

Proof of claim. Write first that

$$\begin{aligned}
&\mathscr{H}_\theta(\chi_{S_R} f)(x,y) \\
&\quad = \frac{1}{\pi}\int_{\mathbb{R}} \chi_{S_R}(x - t\cos\theta, y - t\sin\theta) g(x+y-(\cos\theta+\sin\theta)t)\frac{dt}{t} \\
&\quad = \frac{1}{\pi}\int_{\mathbb{R}} \chi_{S_R}\left(x - s\frac{\cos\theta}{\cos\theta+\sin\theta}, y - s\frac{\sin\theta}{\cos\theta+\sin\theta}\right) g(x+y-s)\frac{ds}{s}.
\end{aligned}$$

To check that this agrees with $\mathscr{H}g(x+y)$, it is sufficient to check that in the last integral χ_{S_R} does not vanish unless g vanishes. Let us thus assume that $g(x+y-s)$ does not vanish for some s. Along with the assumption $(x,y) \in N_R$ we get the inequalities

$$|x+y-s| < k, \quad |x-y| < R - R^{\frac{3}{4}}, \quad |x+y| < \frac{1}{2}R^{\frac{1}{4}} - k,$$

and we are required to prove that

$$|(x-sa)-(y-sb)| < R,$$

where $a = \cos\theta/(\cos\theta + \sin\theta)$ and $b = \sin\theta/(\cos\theta+\sin\theta)$. This follows from the estimate

$$\begin{aligned} |x-y-s(a-b)| &= |x-y+(x+y-s)(a-b)-(x+y)(a-b)| \\ &\leq R - R^{\frac{3}{4}} + (k + \frac{1}{2}R^{\frac{1}{4}} - k)|a-b| \\ &= R + R^{\frac{1}{4}}\left(\frac{|a-b|}{2} - R^{\frac{1}{2}}\right), \end{aligned}$$

where the last term is negative since

$$\frac{|a-b|}{2} = \left|\frac{\cos\theta - \sin\theta}{2(\cos\theta+\sin\theta)}\right| \leq |\cos\theta + \sin\theta|^{-1} < R^{\frac{1}{2}},$$

where we have used the assumption that R was sufficiently large.

This concludes the proof of our claim, and we return to the proof of the main theorem. Let us denote by $T_{\Omega,\epsilon}$ the operator given by

$$T_{\Omega,\epsilon}h = -\pi i \int_{-\pi/4+\epsilon}^{3\pi/4-\epsilon} \Omega(e^{i\theta})\mathscr{H}_\theta(h)(z)d\theta.$$

Note that (2.8) implies that

$$\begin{aligned} &(T_\Omega - T_{\Omega,\epsilon})h \\ &\quad = -\pi i \left(\int_{-\pi/4}^{-\pi/4+\epsilon} \Omega(e^{i\theta})\mathscr{H}_\theta(h)(z)d\theta + \int_{3\pi/4-\epsilon}^{3\pi/4} \Omega(e^{i\theta})\mathscr{H}_\theta(h)(z)d\theta \right), \end{aligned}$$

and from this we see that the operator norm of $T_\Omega - T_{\Omega,\epsilon}$ goes to zero as $\epsilon \to 0$. Now for each $\epsilon > 0$ the expression $|\cos\theta + \sin\theta|^{-1}$ is bounded on $(-\pi/4+\epsilon, 3\pi/4-\epsilon)$, and thus for R large enough we have, by the previous claim, that

$$\begin{aligned} T_{\Omega,\epsilon}(\chi_{S_R}f)(x,y) &= -\pi i \int_{-\pi/4+\epsilon}^{3\pi/4-\epsilon} \Omega(e^{i\theta})\mathscr{H}g(x+y)d\theta \\ &= \left(-\pi i \int_{-\pi/4+\epsilon}^{3\pi/4-\epsilon} \Omega(e^{i\theta})d\theta\right)\mathscr{H}g(x+y) \end{aligned}$$

for all $(x, y) \in N_R$. Denote $C_{\Omega,\epsilon} = -\pi i \int_{-\pi/4+\epsilon}^{3\pi/4-\epsilon} \Omega(e^{i\theta})d\theta$. Now we may estimate

$$\begin{aligned} ||T_{\Omega,\epsilon}||_p^p &\geq \frac{||T_{\Omega,\epsilon}(\chi_{S_R} f)(x,y)||_p^p}{||\chi_{S_R} f||_p^p} \\ &\geq \frac{||\chi_{N_R}(x,y) T_{\Omega,\epsilon}(\chi_{S_R} f)(x,y)||_p^p}{R||g||_p^p} \\ &= \frac{|C_{\Omega,\epsilon}|^p ||\chi_{N_R}(x,y)\mathscr{H} g(x+y)||_p^p}{R||g||_p^p}. \end{aligned}$$

Taking the limit as $R \to \infty$, we find by (2.9) that

$$||T_{\Omega,\epsilon}||_p^p \geq \frac{|C_{\Omega,\epsilon}|^p ||\mathscr{H} g||_p^p}{||g||_p^p}.$$

Taking the supremum over all g in C_0^∞ yields

$$||T_{\Omega,\epsilon}||_p \geq |C_{\Omega,\epsilon}| ||\mathscr{H}||_p,$$

and finally letting ϵ go to zero shows that

$$||T_\Omega||_p \geq |C_\Omega| ||\mathscr{H}||_p, \tag{2.10}$$

where

$$|C_\Omega| = \pi \left| \int_{-\pi/4}^{3\pi/4} \Omega(e^{i\theta})d\theta \right|. \tag{2.11}$$

Note that if we replace the function $\Omega(z)$ by $\Omega(e^{i\phi}z)$ for some angle ϕ, we get an operator of same norm but in this case

$$|C_{\Omega_\phi}| = \pi \left| \int_{-\pi/4+\phi}^{3\pi/4+\phi} \Omega(e^{i\theta})d\theta \right|,$$

which shows that the domain of integration in (2.11) may be replaced by any semicircle. □

For T_Ω equal to the complex Riesz transform we have that

$$|C_\Omega| = \frac{1}{2} \left| \int_{-\pi/4}^{3\pi/4} e^{-i\theta} d\theta \right| = 1.$$

Since the Hilbert transform attains its maximum norm over a real-valued function, g (and hence f) may be taken to be real-valued above. We have thus proven our earlier claim that

$$||\mathcal{R}||_p^{\mathbb{R}} \geq ||\mathscr{H}||_p. \tag{2.12}$$

Notice that if Ω is nonnegative on any semicircle, the inequality (2.10) becomes an equality as we also have the upper estimate

$$||T_\Omega f||_p \leq \frac{\pi||\mathscr{H}||_p}{2}\int_0^{2\pi}|\Omega(e^{i\theta})|d\theta,$$

where we have used Minkowski's integral inequality. Especially the norms of the planar Riesz transforms may be computed as

$$||R_1||_p = \pi||\mathscr{H}||_p\int_{-\pi/2}^{\pi/2}\frac{1}{2\pi}\cos\theta d\theta = ||\mathscr{H}||_p$$

and similarly $||R_2||_p = ||\mathscr{H}||_p$.

We may now also prove our claimed equality $||T_2||_p^{\mathbb{R}} = ||\mathscr{H}||_p$. Let us again consider $f(x,y) = g(x+y)$, where g is compactly supported and real-valued as before. We have for large R and $(x,y) \in N_R$ that $T_2(S_R f)(x,y) = -i\mathscr{H}g(x+y)$, and thus

$$\begin{aligned}\left(||T_2||_p^{\mathbb{R}}\right)^p &\geq \frac{||T_2(\chi_{S_R}f)(x,y)||_p^p}{||\chi_{S_R}f||_p^p}\\ &\geq \frac{||\chi_{N_R}(x,y)T_2(\chi_{S_R}f)(x,y)||_p^p}{R||g||_p^p}\\ &= \frac{||\chi_{N_R}(x,y)\mathscr{H}g(x+y)||_p^p}{R||g||_p^p}.\end{aligned}$$

Letting $R \to \infty$ and taking the supremum over real-valued g shows that $||T_2||_p^{\mathbb{R}} \geq ||\mathscr{H}||_p$, and since we proved the reverse inequality earlier this shows indeed that $||T_2||_p^{\mathbb{R}} = ||\mathscr{H}||_p$.

2.4 A curious identity

We have also found an interesting class of identities for each of our operators T_k. Fix thus k and let ϵ denote the $2k$th root of unity as usual. Recall the decomposition of the complex plane into sectors $S_0, S_1, \ldots, S_{2k-1}$ such that the multiplier m_k of T_k is the constant ϵ^{-j} in each sector S_j. Let us now decompose $L^2(\mathbb{R}^2)$ into $2k$ subspaces given by

$$L_j^2 = \{f \in L^2 : \hat{f}(\xi) = 0 \text{ for all } \xi \in \mathbb{C}\setminus S_j\}.$$

Evidently each L_j^2 is a subspace of L^2, and they are pairwise orthogonal as the Fourier transform preserves the inner product in L^2. We claim that these spaces are also closed under multiplication, i.e. each one forms an algebra. For this choose any f and g in L_j^2. Note that

$$\widehat{(fg)}(\xi) = (\hat{f} * \hat{g})(\xi) = \int_{\mathbb{C}}\hat{f}(z)\hat{g}(\xi - z)dz.$$

The trick now is to notice that the sectors S_j are convex and closed under homothety with respect to the origin, and thus we have the set equality

$$S_j + S_j = \{s_1 + s_2 : s_1, s_2 \in S_j\} = S_j$$

for each j. So if we are given $\xi \in \mathbb{C} \setminus S_j$ and $z \in \mathbb{C}$, it follows that one of z and $\xi - z$ must also lie in $\mathbb{C} \setminus S_j$, as their sum is ξ. Thus for $\xi \in \mathbb{C} \setminus S_j$ we have that

$$\widehat{(fg)}(\xi) = \int_{\mathbb{C}} \hat{f}(z)\hat{g}(\xi - z)dz = \int_{\mathbb{C}} 0 \; dz = 0,$$

which proves that $fg \in L^2_j$ as wanted.

Since the subspaces L^2_j are pairwise orthogonal and span L^2, any function $f \in L^2$ may be decomposed as

$$f = \sum_{j=0}^{2k-1} f_j,$$

where each f_j lies in L^2_j. Now notice that the operator T_k acts on L^2_j simply by multiplying a given function with ϵ^{-j}. It follows that

$$T_k f = \sum_{j=0}^{2k-1} T_k f_j = \sum_{j=0}^{2k-1} \epsilon^{-j} f_j.$$

Iterating this, we obtain

$$\begin{aligned} f &= \sum_{j=0}^{2k-1} f_j \\ T_k f &= \sum_{j=0}^{2k-1} \epsilon^{-j} f_j \\ T_k^2 f &= \sum_{j=0}^{2k-1} \epsilon^{-2j} f_j \\ &\dots \\ T_k^{2k-1} f &= \sum_{j=0}^{2k-1} \epsilon^{-(2k-1)j} f_j. \end{aligned}$$

Now we may express each f_j in terms of T_k and f by the identity

$$\sum_{l=0}^{2k-1} \epsilon^{jl} T_k^l f = \sum_{l,m=0}^{2k-1} \epsilon^{jl} \epsilon^{-lm} f_m = \sum_{m=0}^{2k-1} f_m \sum_{l=0}^{2k-1} \epsilon^{l(j-m)} = 2k f_j,$$

where we have used the property of the $2k$th root of unity that

$$\sum_{l=0}^{2k-1} \epsilon^{ln} = \begin{cases} 2k, & \text{if } 2k|n \\ 0, & \text{otherwise} \end{cases}.$$

We now do one last technical computation before the main identity. Fix any nonnegative integers m and N, and assume that the Nth power of each f_j is also in L^2. Since each L^2_j is an algebra, we may compute that

$$\begin{aligned}
(2k)^N T_k^m \left(\sum_{j=0}^{2k-1} f_j^N \right) &= T_k^m \left(\sum_{j=0}^{2k-1} (2kf_j)^N \right) \\
&= \sum_{j=0}^{2k-1} T_k^m (2kf_j)^N \\
&= \sum_{j=0}^{2k-1} \epsilon^{-jm} (2kf_j)^N \\
&= \sum_{j=0}^{2k-1} \epsilon^{-jm} \left(\sum_l \epsilon^{jl} T_k^l f \right)^N \\
&= \sum_{j=0}^{2k-1} \epsilon^{-jm} \sum_{i_1,\ldots,i_N} \epsilon^{j(i_1+\cdots i_N)} (T_k^{i_1} f) \cdots (T_k^{i_N} f) \\
&= \sum_{i_1,\ldots,i_N} (T_k^{i_1} f) \cdots (T_k^{i_N} f) \sum_{j=0}^{2k-1} \epsilon^{j((i_1+\cdots i_N)-m)} \\
&= 2k \sum_{i_1+\cdots+i_N \equiv m} (T_k^{i_1} f) \cdots (T_k^{i_N} f),
\end{aligned}$$

where the last summation runs over all ordered sequences $(i_1, \ldots, i_N)$ of integers in $[0, 2k-1]$ for which $i_1 + \cdots + i_N \equiv m \pmod{2k}$. Plugging in $m = 0$ we obtain

$$(2k)^N \sum_{j=0}^{2k-1} f_j^N = 2k \sum_{i_1+\cdots+i_N \equiv 0} (T_k^{i_1} f) \cdots (T_k^{i_N} f).$$

Plugging this back into the previous calculation we arrive at our identity

$$T_k^m \left(\sum_{i_1+\cdots+i_N \equiv 0} (T_k^{i_1} f) \cdots (T_k^{i_N} f) \right) = \sum_{i_1+\cdots+i_N \equiv m} (T_k^{i_1} f) \cdots (T_k^{i_N} f). \quad (2.13)$$

A simple way to think of this equation is to introduce a dummy variable X for which $X^{2k} = 1$. We say that a formal sum of functions

$$\sum_{l=0}^{2k-1} F_l X^l$$

is a T_k-sum, if $F_l = T_k^l F_0$ for each l. Then (2.13) says that any power of a T_k-sum is again a T_k-sum. It is also possible to generalize the above argument to show that products of T_k-sums are again T_k-sums, i.e. that the identity

$$T_k^m \left(\sum_{i_1+\cdots+i_N \equiv 0} (T_k^{i_1} f^{(1)}) \cdots (T_k^{i_N} f^{(N)}) \right) = \sum_{i_1+\cdots+i_N \equiv m} (T_k^{i_1} f^{(1)}) \cdots (T_k^{i_N} f^{(N)})$$

holds for any N sufficiently well-behaved functions $f^{(1)}, \ldots, f^{(N)}$. The equation (2.13) contains quite a lot of identities in it. For example, setting $k = m = N = 1$ we obtain

$$T_1(f^2 + (T_1 f)^2) = 2f T_1 f,$$

which, in accordance with the definition $T_1 = i\mathscr{H}_{\pi/2}$, is equivalent to a well-known identity for the Hilbert transform,

$$f^2 + (\mathscr{H} f)^2 = 2\mathscr{H}(f \mathscr{H} f).$$

In fact, a classical proof of the boundedness of $\mathscr{H}$ on the L^p-spaces relies on this identity to find (from the L^2-boundedness of $\mathscr{H}$) a bound on $||\mathscr{H}||_p$ for each p of the form $p = 2^n$, n a positive integer. The p-norm estimates obtained in this way turn out to be optimal for $p = 2^n$, see (1.1) for these exact norms. Unfortunately for us, the rest of our identities do not seem to provide any bounds on the p-norms of the higher order operators T_k, $k \geq 2$.

3 Fredholm theory

In this short section we recall some results on the Fredholm index from functional analysis. The results we need are not too difficult and thus they are stated without proof. A more curious reader is referred to the book on functional analysis by John B. Conway, [Con90], where the facts are proven for Hilbert spaces in a way not too difficult to generalize to our setting.

We assume throughout the section that X and Y are Banach spaces. Let $\mathcal{B}(X,Y)$ denote the space of bounded operators between X and Y, where we always assume that an operator is linear (with respect to the given scalar field). Recall that a subset of a topological space is said to be relatively compact if its closure is compact. We then define that an operator $T : X \mapsto Y$ is compact if it maps the unit ball (or equivalently every bounded set) in X to a relatively compact set in Y. One of the immediate results is that the compact operators form a closed ideal in the algebra $\mathcal{B}(X,Y)$.

A bounded operator $A : X \mapsto Y$ is called *left semi-Fredholm* if there exists a bounded operator B and a compact operator K such that $BA = 1 + K$, where 1 denotes the identity operator. Right semi-Fredholm operators are defined analogously, and Fredholm operators are those which are both left and right semi-Fredholm. We let $\mathcal{F}(X,Y) \subset \mathcal{B}(X,Y)$ denote the space of Fredholm operators from X to Y. There is an alternate characterization of Fredholm operators: They are those bounded operators whose kernel and cokernel are both finite-dimensional. Thus for a Fredholm operator T we may define the *Fredholm index* by

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T.$$

The main result we will need is the following continuity property.

Theorem 3.1. *The map $T \mapsto \operatorname{ind} T$ is a continuous map from the space $\mathcal{F}(X,Y)$ of Fredholm operators to the integers $\mathbb{Z}$ equipped with the discrete topology, i.e. it is constant in each component of $\mathcal{F}(X,Y)$.*

4 The Beltrami equation

Let us define for a C^1-function $f : \mathbb{C} \mapsto \mathbb{C}$ the Cauchy-Riemann complex derivatives f_z and $f_{\overline{z}}$ by

$$f_z = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad f_{\overline{z}} = \frac{1}{2}(f_x + i f_y).$$

The naming stems from the fact that the equation $f_{\overline{z}} = 0$ is equivalent with the Cauchy-Riemann equations for holomorphic functions.

In complex function theory we are often concerned about mapping properties of given or unknown functions. One of these properties is the notion of quasiconformality, which in a nutshell means that our function f maps small circles to ellipses with a uniform bound on the ratio between the longer and shorter axes. As our mapping is of C^1-class this condition may be written as the distortion inequality

$$\frac{\max_{|v|=1} |Df(z)v|}{\min_{|v|=1} |Df(z)v|} \leq K < \infty, \tag{4.1}$$

where K is not dependant on the chosen point z. In terms of complex derivatives one sees that $\max_{|v|=1} |Df(z)v| = |f_z(z)| + |f_{\overline{z}}(z)|$ and $\min_{|v|=1} |Df(z)v| = ||f_z(z)| - |f_{\overline{z}}(z)||$. Assuming further that our mapping f is orientation-preserving leads to $J_f(z) = \det Df(z) = |f_z|^2 - |f_{\overline{z}}|^2 \geq 0$, and hence the inequality (4.1) takes the form

$$\frac{|f_z(z)| + |f_{\overline{z}}(z)|}{|f_z(z)| - |f_{\overline{z}}(z)|} \leq K < \infty \quad \text{or} \quad \frac{|f_{\overline{z}}(z)|}{|f_z(z)|} \leq \frac{K-1}{K+1} < 1.$$

We may now define more generally that a homeomorphism $f : \Omega \mapsto \Omega'$ between planar domains is quasiconformal if it has distributional partial derivatives locally in L^2 and its complex derivatives satisfy the almost everywhere bound

$$|f_{\overline{z}}| \leq k|f_z|$$

for some $k < 1$. This condition is equivalent with f solving the so-called Beltrami equation

$$f_{\overline{z}} = \mu f_z, \tag{4.2}$$

where μ, called the complex dilatation of f, is some measurable function that satisfies the condition $||\mu(z)||_\infty < 1$. This geometrically pleasing description of the Beltrami equation should be enough to motivate the attention we will give it in the future. But it should also be mentioned that the study of this equation is of fundamental importance in quasiconformal geometry. For an introduction to the subject see the classical book by Ahlfors, [Ahl66]. We will instead constrain ourselves to the inhomogenous equation

$$f_{\overline{z}} - \mu f_z = h,$$

whose solvability question we will address using theory that should be familiar to the reader by now. Indeed, as with differential equations in general, this question will be reduced to the invertibility of a certain operator. In this case the operator will involve a singular integral we have already met before, the Beurling transform.

Let us thus begin a more rigorous study of the Beurling transform, which we define originally only for functions in C_0^∞ via the principal value integral

$$\mathcal{S}f(z) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|z-\omega|>\epsilon} \frac{f(\omega)}{(z-\omega)^2} d\omega.$$

Our first concern is to check that this coincides with the Fourier multiplier definition given before in Section 2. For $f \in C_0^\infty$ we may use Green's formula to compute that

$$\begin{aligned} \mathcal{S}f_{\overline{z}}(z) &= -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|z-\omega|>\epsilon} \frac{f_{\overline{z}}(\omega)}{(z-\omega)^2} d\omega \\ &= \frac{i}{2\pi} \lim_{\epsilon \to 0} \int_{\partial B(z,\epsilon)} \frac{f(\omega)}{(z-\omega)^2} d\omega \\ &= -\frac{i}{2\pi} \lim_{\epsilon \to 0} \frac{\partial}{\partial z} \int_{\partial B(z,\epsilon)} \frac{f(\omega)}{z-\omega} d\omega, \end{aligned}$$

where we have disregarded the fact that the domains $\mathbb{C} \setminus B(z,\epsilon)$ are unbounded since f vanishes at a neighbourhood of infinity. Another application of Green's formula shows that

$$
\begin{aligned}
-\frac{i}{2\pi}\lim_{\epsilon\to 0}\frac{\partial}{\partial z}\int_{\partial B(z,\epsilon)}\frac{f(\omega)}{z-\omega}d\omega &= \lim_{\epsilon\to 0}\frac{\partial}{\partial z}\left(f(z)+\frac{1}{\pi}\int_{B(z,\epsilon)}\frac{f_{\overline{z}}(\omega)}{z-\omega}d\omega\right)\\
&= f_z(z)+\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{B(0,\epsilon)}\frac{f_{\overline{z}z}(z+\omega)}{\omega}d\omega\\
&= f_z(z),
\end{aligned}
$$

and hence $\mathcal{S}f_{\overline{z}} = f_z$.

In view of the identities $\widehat{(f_z)}(\xi) = \pi i\overline{\xi}\widehat{f}(\xi)$ and $\widehat{(f_{\overline{z}})}(\xi) = \pi i\xi\widehat{f}(\xi)$ we have just proven that

$$
\widehat{\mathcal{S}f_{\overline{z}}} = \frac{\overline{\xi}}{\xi}\widehat{f_{\overline{z}}}(\xi) = \frac{\overline{\xi}^2}{|\xi|^2}\widehat{f_{\overline{z}}}(\xi)
$$

as expected. It is also not a problem that we have only verified this multiplier identity for functions of the form $f_{\overline{z}}$, where $f \in C_0^\infty$. Indeed, if g lies in the L^2-orthogonal complement of $\{f_{\overline{z}} : f \in C_0^\infty\}$, then $\int f_{\overline{z}}\overline{g} = 0$ for all $f \in C_0^\infty$, and hence $\overline{g}$ satisfies the equation $\overline{g}_{\overline{z}} = 0$ in the weak sense. But then Weyl's lemma implies that $\overline{g}$ is holomorphic. The only holomorphic function in L^2 is the constant zero, which implies that $g = 0$ and hence that $\{f_{\overline{z}} : f \in C_0^\infty\}$ is dense in L^2. Thus we find that

$$
\widehat{\mathcal{S}g}(\xi) = \frac{\overline{\xi}^2}{|\xi|^2}\widehat{g}(\xi)
$$

for all $g \in C_0^\infty$.

It now immediately follows that $\mathcal{S}$ is an L^2-isometry on C_0^∞ and thus may be uniquely extended to an isometry on the whole L^2. As we have proven before, $\mathcal{S}$ may also be extended to be bounded on L^p for $1 < p < \infty$. The multiplier of $\mathcal{S}$ is even, which further implies the identity $\mathcal{S}^{-1}f = \overline{\mathcal{S}\overline{f}}$ and the fact that $\mathcal{S}$ is symmetric, i.e.

$$
\int_{\mathbb{C}} \mathcal{S}f(z)g(z)dz = \int_{\mathbb{C}} f(z)\mathcal{S}g(z) \tag{4.3}
$$

for $f, g \in C_0^\infty$. By continuity we may extend this identity to $f \in L^p$ and $g \in L^q$, where p and q are Hölder conjugates.

Let us now assume that f is locally integrable and that it has distributional partial derivatives in L^p. Now for any $g \in C_0^\infty$,

$$
\int_{\mathbb{C}}(\mathcal{S}f_{\overline{z}})g = \int_{\mathbb{C}} f_{\overline{z}}\mathcal{S}g = -\int_{\mathbb{C}} f\partial_{\overline{z}}\mathcal{S}g = -\int_{\mathbb{C}} fg_z = \int_{\mathbb{C}} f_z g.
$$

Note that the second equality does not immediately follow from the definition of distributional derivatives, as $\mathcal{S}g$ may lack compact support and thus might not be a valid test function. To justify this equality one should approximate $\mathcal{S}g$ with a suitable sequence of C_0^∞-functions, but we will trust the reader to fill in the details.

The previous computation now shows that the identity

$$\mathcal{S}f_{\overline{z}} = f_z \tag{4.4}$$

also holds when the complex derivatives are taken in the distributional sense with the same assumptions as above on f.

Theorem 4.1. *Let us define the* maximal Beurling transform $\mathcal{S}_*$ *by*

$$\mathcal{S}_* f(x) = \sup_{\epsilon>0} \left| \frac{1}{\pi} \int_{|z-\omega|>\epsilon} \frac{f(\omega)}{(z-\omega)^2} d\omega \right| . \tag{4.5}$$

Then $\mathcal{S}_* : L^p \mapsto L^p$ is a bounded map.

Proof. There is a surprisingly simple proof available, found not too long ago by Mateu and Verdera in [MV06]. First we need to calculate the Beurling transform of a particular function, namely the characteristic function of a disc $B(a,r)$. In view of (4.4) it suffices to find a nice enough function g such that $g_{\overline{z}} = \chi_{B(a,r)}$. It is not hard to come up with

$$g(z) = \begin{cases} \overline{z} - \overline{a}, & z \in B(a,r) \\ r^2/(z-a), & z \notin B(a,r) \end{cases} ,$$

which we may then use to calculate that

$$\mathcal{S}\chi_{B(a,r)} = g_z = -\frac{r^2}{(z-a)^2}\chi_{\mathbb{C}\setminus B(a,r)}.$$

This and (4.3) let us produce the remarkable identity

$$-\frac{1}{\pi} \int_{|z-\omega|>\epsilon} \frac{f(\omega)}{(z-\omega)^2} d\omega = \frac{1}{\pi\epsilon^2} \int_{\mathbb{C}} \mathcal{S}(\chi_{B(z,\epsilon)}) f = \frac{1}{|B(z,\epsilon)|} \int_{B(z,\epsilon)} \mathcal{S}f,$$

from which it is immediately clear that

$$\mathcal{S}_* f(z) \leq \mathcal{M}\mathcal{S}f(z),$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function, which is known to be bounded on L^p for $1 < p < \infty$. Thus the boundedness of $\mathcal{S}_*$ follows from the already proven boundedness of $\mathcal{S}$. □

Theorem 4.2. *Let $\mu \in C_0^\infty$. Then the operator $[\mu, \mathcal{S}] = \mu\mathcal{S} - \mathcal{S}\mu$ is compact from L^p to itself, where $1 < p < \infty$.*

Proof. We first refer to the Kolmogorov-Riesz (or Frechet-Kolmogorov) theorem, which characterizes relative compactness in L^p-spaces. A set $\mathcal{F}$ is relatively compact in $L^p(\mathbb{R}^n)$ if and only if the following three properties are fulfilled

1. $\mathcal{F}$ is bounded in $L^p(\mathbb{R}^n)$.
2. For every $\epsilon > 0$ there exists $R > 0$ so that for all $f \in \mathcal{F}$
$$\left(\int_{|x|>R} |f(x)|^p dx \right)^{\frac{1}{p}} < \epsilon.$$
3. For every $\epsilon > 0$ there exists $\delta > 0$ so that for all $f \in \mathcal{F}$ and $h \in \mathbb{R}^n$ with $|h| < \delta$:
$$\left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx \right)^{\frac{1}{p}} < \epsilon.$$

Our set of interest is of course the image of the unit ball in L^p under the operator $[\mu, \mathcal{S}]$. To prove these three properties for this set we follow a framework laid out by A. Uchiyama in [Uch78]. Let us first choose positive constants R, K and L such that $|\mu(z)| \leq K\chi_{B(0,R)}$ and $|\mu(z+h) - \mu(z)| \leq L|h|$ for all $z, h \in \mathbb{C}$. We notice that the first condition readily follows from the boundedness of $\mathcal{S}$, and for the second one it suffices to estimate that for $|z| > 2R$:

$$\begin{aligned}
|[\mu, \mathcal{S}]f(z)| &= \frac{1}{\pi} \left| \text{p.v.} \int \frac{\mu(z) - \mu(\omega)}{(z-\omega)^2} f(\omega) d\omega \right| \\
&= \frac{1}{\pi} \left| \int_{|\omega|<R} \frac{\mu(\omega)}{(z-\omega)^2} f(\omega) d\omega \right| \\
&\leq \frac{K}{\pi(|z|-R)^2} \int_{|\omega|<R} |f(\omega)| d\omega \\
&\leq C_1 |z|^{-2} ||f||_p.
\end{aligned}$$

The proof of the third condition is a little more involved. Let $\epsilon > 0$, and notice first the identity

$$
\begin{aligned}
&[\mu,\mathcal{S}]f(z+h)-[\mu,\mathcal{S}]f(z)\\
&\quad=\frac{1}{\pi}\text{p.v.}\int\frac{\mu(z)-\mu(\omega)}{(z-\omega)^2}f(\omega)d\omega\\
&\quad\quad-\frac{1}{\pi}\text{p.v.}\int\frac{\mu(z+h)-\mu(\omega)}{(z+h-\omega)^2}f(\omega)d\omega\\
&\quad=-\frac{1}{\pi}\int_{|z-\omega|>\epsilon^{-1}|h|}\frac{\mu(z+h)-\mu(z)}{(z-\omega)^2}f(\omega)d\omega\\
&\quad\quad-\frac{1}{\pi}\int_{|z-\omega|>\epsilon^{-1}|h|}(\mu(\omega)-\mu(z+h))\left(\frac{1}{(z-\omega)^2}-\frac{1}{(z+h-\omega)^2}\right)f(\omega)d\omega\\
&\quad\quad+\frac{1}{\pi}\text{p.v.}\int_{|z-\omega|<\epsilon^{-1}|h|}\frac{\mu(z)-\mu(\omega)}{(z-\omega)^2}f(\omega)d\omega\\
&\quad\quad-\frac{1}{\pi}\text{p.v.}\int_{|z-\omega|<\epsilon^{-1}|h|}\frac{\mu(z+h)-\mu(\omega)}{(z+h-\omega)^2}f(\omega)d\omega.
\end{aligned}
$$

Now the first term is bounded by

$$L|h|\mathcal{S}_*f(z),$$

whose L^p norm goes to zero at a rate independent of f due to Theorem 4.1.

The second term is dominated by

$$C_2|h|\int_{|z-\omega|>\epsilon^{-1}|h|}|z-\omega|^{-3}|f(\omega)|d\omega.$$

We may estimate the L^p-norm of this expression with Minkowski's integral inequality:

$$
\begin{aligned}
C_2|h|&\left(\int_{\mathbb{C}}\left(\int_{|\omega|>\epsilon^{-1}|h|}|\omega|^{-3}|f(z+\omega)|d\omega\right)^p dz\right)^{\frac{1}{p}}\\
&\leq C_2|h|\int_{|\omega|>\epsilon^{-1}|h|}|\omega|^{-3}\left(\int_{\mathbb{C}}|f(z+\omega)|^p dz\right)^{\frac{1}{p}}d\omega\\
&=C_2|h|||f||_p\int_{|\omega|>\epsilon^{-1}|h|}|\omega|^{-3}d\omega\\
&=C_3\epsilon||f||_p.
\end{aligned}
$$

In the same vein we first bound the third term by

$$\frac{L}{\pi}\int_{|z-\omega|<\epsilon^{-1}|h|}|z-\omega|^{-1}|f(\omega)|d\omega,$$

and do essentially the same integration to obtain a bound of

$$C_4\epsilon^{-1}|h|||f||_p.$$

This technique also applies to the fourth term after changing the domain of integration a little. Indeed, after the initial estimate of passing absolute values inside the integral we may enlarge the domain of integration to the set $\{\omega : |z+h-\omega| < (1+\epsilon^{-1})|h|\}$ and thus obtain the bound of

$$C_4(1+\epsilon^{-1})|h|||f||_p.$$

Taking $|h|$ small enough we finally find that

$$||[\mu, \mathcal{S}]f(z+h) - [\mu, \mathcal{S}]f(z)||_p \leq C_5\epsilon||f||_p,$$

from which we may conclude that the third condition holds, so $[\mu, \mathcal{S}]$ is compact. □

This theorem also has several extensions. One might for example want to replace $\mathcal{S}$ with other operators. The case of a general Calderón-Zygmund operator is handled in [Uch78]. Another operator we will need this result for is $\mathcal{S}^{-1}$, for which it easily follows from the identity $\mathcal{S}^{-1}f = \overline{\mathcal{S}\overline{f}}$ and the fact that $f \mapsto \overline{f}$ is an isometric homeomorphism of L^p for any p. Another way to generalize is to get rid of some of the assumptions on μ by a density argument. The right space to consider happens to be BMO, the space of functions of bounded mean oscillation, i.e. of bounded BMO-norm

$$||\mu||_{BMO} = \sup_D \frac{1}{|D|}\int_D |\mu - \mu_D|,$$

where the supremum is taken over all discs $D \subset \mathbb{C}$ and $\mu_D = 1/|D|\int_D \mu$. Usually we also quotient out by constant functions, so that BMO will become a Banach space. The following theorem is due to Coifman, Rochberg and Weiss, [CRW76].

Theorem 4.3. *Let $\mu \in BMO$. Then $[\mathcal{S}, \mu]$ is bounded on L^p for $1 < p < \infty$ with*

$$||[\mathcal{S}, \mu]||_p \leq A_p||\mu||_{BMO},$$

where A_p only depends on p.

Let VMO denote the completion of the space C_0^∞ under the BMO-norm. Then given $\mu \in VMO$ we may choose a sequence $(\mu_n) \subset C_0^\infty$ converging to μ in BMO. The preceding theorem shows that

$$||[\mathcal{S}, \mu] - [\mathcal{S}, \mu_n]||_p = ||[\mathcal{S}, \mu - \mu_n]||_p \leq A_p||\mu - \mu_n||_{BMO}.$$

It follows that $[\mathcal{S}, \mu_n] \to [\mathcal{S}, \mu]$ in L^p as $n \to \infty$. Since the compact operators form a closed subspace, we conclude that $[\mathcal{S}, \mu]$ is compact. This holds in particular when μ is continuous and vanishes at infinity.

4.1 Solving the equations

Let us now show the fruits of the preceding computations. Consider again the differential equation $f_{\overline{z}} - \mu f_z = h$. We use the method presented in [AIM09] to show the existence and uniqueness of a solution in some sense, utilizing both the previous theorem and the results on the Fredholm index. We assume that $||\mu||_\infty < 1$ with the additional condition that $\mu \in C_0$. Recall that the condition $||\mu||_\infty < 1$ was imposed because of the relation to quasiconformal mappings. This condition is also related to the notion of ellipticity for partial differential operators which we explore more precisely later. The differential operator $\partial_{\overline{z}} - \mu \partial_z$ happens to be (uniformly) elliptic exactly when we have the almost everywhere bound $|\mu| \leq k < 1$ or (by symmetry) $|\mu| \geq k > 1$.

We will look for solutions in the L^p-setting, so we take h to be in L^p for some p, assuming $p \geq 2$ for the moment. Let us denote $f_{\overline{z}} = \omega$, so that f may be reconstructed up to an analytic function if we know ω. Using identity (4.4) the associated equation for ω becomes $\omega - \mu \mathcal{S} \omega = h$, where we recall the Beurling transform $\mathcal{S}$. We shall look for solutions in L^p so that the question reduces to the invertibility of the operator $1 - \mu\mathcal{S}$. The usual method of Neumann series seems inviting, but there is no guarantee that $||\mu \mathcal{S}||_p < 1$. Instead we will settle for the partial sums $P_n = \sum_{k=0}^{n} (\mu \mathcal{S})^k$ of the Neumann series for μS. These partial sums should at least be good approximations to an inverse of $1 - \mu\mathcal{S}$. Indeed,

$$P_{n-1}(1 - \mu\mathcal{S}) = (1 - \mu\mathcal{S})P_{n-1} = 1 - (\mu\mathcal{S})^n = 1 - \mu^n \mathcal{S}^n + K,$$

where K can be written as a finite sum of products containing the commutator $[\mu, \mathcal{S}]$, and thus is compact by a combination of Theorem 4.2 and the fact that the set of compact operators is an ideal. We deduce that $1 - \mu\mathcal{S}$ is a Fredholm operator if $1 - \mu^n \mathcal{S}^n$ is invertible for some n. For the invertibility we again apply the Neumann series, so that it is enough to prove that

$$||\mu^n \mathcal{S}^n||_p \to 0 \quad \text{as } n \to \infty. \tag{4.6}$$

The estimate $||\mu^n \mathcal{S}^n||_p \leq ||\mu||_\infty^n ||\mathcal{S}^n||_p$ is imminent, but there is no obvious way to see that the norms of $\mathcal{S}^n$ grow slower than exponentially. This is, however, an immediate consequence of the following theorem stated as Corollary 4.5.1 in [AIM09].

Theorem 4.4. *Let $1 < p < \infty$. Then for any $n \in \mathbb{Z}$ the nth iterate of the complex Riesz transform satisfies:*

$$||\mathcal{R}^n||_p \leq C_p(1 + n^2),$$

where C_p only depends on p.

But we also remark that there is another relatively simple proof of (4.6) using the spectral radius theorem as soon as the spectral radius of $\mathcal{S}$ is known to equal one.

Thus $1 - \mu\mathcal{S}$ is shown to be Fredholm, hence invertibility becomes equivalent to the fact that both the kernel and the Fredholm index are zero. To compute the index we refer to Theorem 3.1, i.e. the continuity of the Fredholm index. For $0 \leq t \leq 1$ we may replace μ by $t\mu$ above to find that the operator $1 - t\mu\mathcal{S}$ is Fredholm. This gives a continuous path between $1 - \mu S$ and the identity operator, which shows that the index is indeed zero.

To prove that the kernel is trivial we first note that if $p \geq 2$ and ω is a solution in L^p to the equation

$$\omega = \mu\mathcal{S}\omega,$$

then ω has compact support since μ does. Thus ω is also in L^2. But $\mathcal{S}$ is an isometry in L^2 so we obtain

$$||\omega||_2 = ||\mu\mathcal{S}\omega||_2 \leq ||\mu||_\infty ||\omega||_2.$$

Since $||\mu||_\infty < 1$ we must have $||\omega||_2 = 0$, so that $\omega = 0$ a.e. and we are done.

We now know that $1 - \mu\mathcal{S}$ is invertible on L^p whenever $p \geq 2$. Note that then also

$$1 - \mathcal{S}\mu = \mathcal{S}(1 - \mu\mathcal{S})\mathcal{S}^{-1}$$

is invertible on L^p since $\mathcal{S}$ is.

There is no reason why we should restrict ourselves to the case $p \geq 2$. Let therefore $1 < q \leq 2$ and denote by p its Hölder conjugate. We use a type of duality argument, which will of course be based on the Riesz representation theorem (Theorem 6.16 in [Rud66]) which says that L^p and L^q are dual spaces of each other. To solve the equation

$$(1 - \mu\mathcal{S})\omega = h$$

in L^q it is enough to find ω such that

$$\langle f, (1 - \mu\mathcal{S})\omega \rangle = \langle f, h \rangle \tag{4.7}$$

holds for every $f \in L^p$, where we have denoted by $\langle f_1, f_2 \rangle = \int_{\mathbb{C}} f_1 f_2$ the duality pairing. Consider the continuous linear functional on L^p given by

$$f \mapsto \left\langle (1 - \mathcal{S}\mu)^{-1} f, h \right\rangle.$$

As we have seen, $1-\mathcal{S}\mu$ is invertible on L^p so by the open mapping theorem the inverse is bounded. Hölder's inequality now shows that this linear functional is also continuous, so there exists some $\omega \in L^q$ so that

$$\big\langle (1-\mathcal{S}\mu)^{-1} f, h\big\rangle = \langle f, \omega\rangle \tag{4.8}$$

holds for all $f \in L^p$. Let us check that ω satisfies (4.7). Using the symmetry properties of $\mathcal{S}$ and the identity (4.8), we see that

$$\begin{aligned}\langle f, (1-\mu\mathcal{S})\omega\rangle &= \langle f, \omega\rangle - \langle \mu f, \mathcal{S}\omega\rangle \\ &= \langle f, \omega\rangle - \langle \mathcal{S}\mu f, \omega\rangle \\ &= \langle (1-\mathcal{S}\mu) f, \omega\rangle \\ &= \big\langle (1-\mathcal{S}\mu)^{-1}(1-\mathcal{S}\mu) f, h\big\rangle \\ &= \langle f, h\rangle\end{aligned}$$

as wanted. We conclude that $1-\mu\mathcal{S}$ is also invertible on L^q. Let us summarize the previous results in the form of the following theorem.

Theorem 4.5. *Let μ be a C_0-function satisfying the uniform bound $|\mu| \leq k < 1$ and h be a given function in L^p for some $1 < p < \infty$. Then the equation*

$$f_{\overline{z}} = \mu f_z + h$$

admits a solution f with $f_{\overline{z}} \in L^p$ that is unique up to an additive constant.

To avoid confusion about originality we stress that this theorem and its proof as presented above are taken from the book [AIM09]. However, for us this result acts as a stepping stone to its generalizations, and as far as our knowledge is concerned, the results that follow are our own.

We will now consider the more general Beltrami equation given as

$$f_{\overline{z}} - \mu f_z - \nu \overline{f_z} = h,$$

where μ and ν are in C_0 with the uniform bound $|\mu(z)| + |\nu(z)| \leq k < 1$. The reason for this form is that it captures the general quasilinear equation $f_{\overline{z}} = H(z, f_z)$, where H is real linear in the second component. The expression $|\mu(z)| + |\nu(z)|$ also happens to be the norm of the linear transformation $\xi \mapsto H(z,\xi) = \mu(z)\xi + \nu(z)\overline{\xi}$, which is easy to verify.

We assume again $h \in L^p$ for $1 < p < \infty$. The same substitution $\omega = f_{\overline{z}}$ reduces this to the invertibility of $1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}}$, where $\overline{\mathcal{S}}$ denotes the operator given by $\overline{\mathcal{S}} f = \overline{\mathcal{S} f}$. We prove the following theorem.

Theorem 4.6. *Let μ and ν be as before and $1 < p < \infty$. Then the ($\mathbb{R}$-linear) operator $1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}}$ is invertible on L^p, and hence for each $h \in L^p$ the equation*

$$f_{\overline{z}} - \mu f_z - \nu \overline{f_z} = h$$

admits a solution f with $f_{\overline{z}} \in L^p$ that is unique up to an additive constant.

It should be remarked that later we will prove an analogue of this theorem in a higher dimensional setting. The proof will actually be shorter but the assumption $|\mu|+|\nu| \leq k < 1$ will have to be tightened to $||\mu||_\infty + ||\nu||_\infty < 1$.

Proof. Let us begin with the case $p \geq 2$. Note that $\overline{\mathcal{S}}$ fails to be a $\mathbb{C}$-linear operator since it is conjugate-linear instead of linear (i.e. $\overline{\mathcal{S}}\alpha f = \overline{\alpha}\overline{\mathcal{S}}f$ for scalars α). This motivates us to look for a right inverse to $1-\mu\mathcal{S}-\nu\overline{\mathcal{S}}$ in the form

$$h \mapsto Th + R\overline{h},$$

where T and R are unknown operators on L^p. This is equivalent with

$$\begin{aligned} &(1-\mu\mathcal{S}-\nu\overline{\mathcal{S}})(Th+R\overline{h}) = h \\ \Leftrightarrow\ &(1-\mu\mathcal{S})Th + (1-\mu\mathcal{S})R\overline{h} - \nu\overline{\mathcal{S}}Th - \nu\overline{\mathcal{S}}R\overline{h} = h. \end{aligned}$$

The linear and conjugate-linear terms should be treated seperately so that we get the system of equations

$$\begin{cases} (1-\mu\mathcal{S})R\overline{h} - \nu\overline{\mathcal{S}}Th &= 0 \\ (1-\mu\mathcal{S})Th - \nu\overline{\mathcal{S}}R\overline{h} &= h \end{cases}.$$

From the first equation we find that $R\overline{h} = (1-\mu\mathcal{S})^{-1}\nu\overline{\mathcal{S}}Th$ and plugging this in the second one gives

$$(1-\mu\mathcal{S}-\nu\overline{\mathcal{S}}(1-\mu\mathcal{S})^{-1}\nu\overline{\mathcal{S}})Th = h,$$

so that the existence of T and R is reduced to the L^p-invertibility of the operator

$$M = 1-\mu\mathcal{S}-\nu\overline{\mathcal{S}}(1-\mu\mathcal{S})^{-1}\nu\overline{\mathcal{S}}.$$

Now due to the identities

$$\overline{\mathcal{S}}f = \mathcal{S}^{-1}\overline{f} \quad \text{and} \quad \overline{(1-\mu\mathcal{S})^{-1}f} = (1-\overline{\mu}\mathcal{S}^{-1})^{-1}\overline{f}$$

we may calculate the conjugates to get

$$M = 1-\mu\mathcal{S}-\nu\mathcal{S}^{-1}(1-\overline{\mu}\mathcal{S}^{-1})^{-1}\overline{\nu}\mathcal{S}.$$

Let us prove that M is Fredholm. First of all since

$$M = 1-\mu\mathcal{S}-\nu(1-\mathcal{S}^{-1}\overline{\mu})^{-1}\overline{\nu}+\nu\mathcal{S}^{-1}(1-\overline{\mu}\mathcal{S}^{-1})^{-1}[\mathcal{S},\overline{\nu}]$$

we might as well ask if

$$M_2 = 1-\mu\mathcal{S}-\nu(1-\mathcal{S}^{-1}\overline{\mu})^{-1}\overline{\nu}$$

is Fredholm. We start by proving that M_2 is right semi-Fredholm. Let

$$M_3 = M_2(1-\mathcal{S}^{-1}\overline{\mu}) = 1-\mu\mathcal{S}-\mathcal{S}^{-1}\overline{\mu}+\mu\overline{\mu}-\nu\overline{\nu}+K_1,$$

where $K_1 = \nu(1-\mathcal{S}^{-1}\overline{\mu})^{-1}[1-\mathcal{S}^{-1}\overline{\mu},\overline{\nu}]$ is compact. If now $\tau = 1+\mu\overline{\mu}-\nu\overline{\nu}$ then

$$|\tau| \geq 1 - |\mu|^2 - |\nu|^2 \geq 1-k^2 > 0$$

holds everywhere and hence $1/\tau$ is well-defined and continuous. We now show that

$$M_4 = (M_3 - K_1)\frac{1}{\tau} = 1 - (\mu\mathcal{S} + \mathcal{S}^{-1}\overline{\mu})\frac{1}{\tau}$$

is Fredholm by a same type of Neumann series argument as for the invertibility of $1-\mu\mathcal{S}$. Fix thus n sufficiently large, and we are to show that the operator

$$1 - \left((\mathcal{S}\mu + \overline{\mu}\mathcal{S}^{-1})\frac{1}{\tau}\right)^n$$

is invertible. Note that commutators of $\mathcal{S}$ and $\mathcal{S}^{-1}$ with μ, and $\overline{\mu}$ are compact by Theorem 4.2. The commutators with $1/\tau$ are also compact as $1/\tau - 1$ is in C_0. Consequently we may write

$$\left((\mathcal{S}\mu + \overline{\mu}\mathcal{S}^{-1})\frac{1}{\tau}\right)^n = M_5 + K_2,$$

where K_2 is compact and M_5 contains the terms after the binomial expansion rearranged into the form

$$\frac{1}{\tau^n}\mu^m\overline{\mu}^{n-m}\mathcal{S}^N,$$

where $|N| \leq n$. There are 2^n of these terms and by Theorem 4.4 we may dominate the norm of each term by

$$||(1/\tau^n)\mu^m\overline{\mu}^{n-m}\mathcal{S}^N||_p \leq ||\mu^m\overline{\mu}^{n-m}/\tau^n||_\infty C_p(1+4N^2) \leq |||\mu|/\tau||_\infty^n C_p(1+4n^2).$$

This leads to the estimate

$$||M_5||_p < C_p(1+4n^2)2^n|||\mu|/\tau||_\infty^n.$$

It remains to show that $2|||\mu|/\tau||_\infty < 1$ so that this will decay to zero. But

$$\begin{aligned}
\frac{2|\mu|}{\tau} &= \frac{2|\mu|}{1+|\mu|^2-|\nu|^2} \\
&\leq \frac{2|\mu|}{1+|\mu|^2-(k-|\mu|)^2} \\
&= \frac{2|\mu|}{1-k^2+2k|\mu|} \\
&\leq \frac{2k}{1+k^2} \\
&< 1
\end{aligned}$$

since $|\mu| + |\nu| \leq k < 1$. This shows that M_2 is right semi-Fredholm. The fact that it is left semi-Fredholm is analogous. Indeed, note that

$$M_6 = (1 - \mathcal{S}^{-1}\overline{\mu})M_2 = 1 - \mu\mathcal{S} - \mathcal{S}^{-1}\overline{\mu} + |\mu|^2 - |\nu|^2 + K_3,$$

where $K_3 = [\nu, 1 - \overline{\mu}\mathcal{S}^{-1}](1 - \overline{\mu}\mathcal{S}^{-1})^{-1}\overline{\nu} + \mathcal{S}^{-1}[\overline{\mu}\mu, \mathcal{S}]$ is again compact. Now note that

$$\frac{1}{\tau}(M_6 - K_3) = 1 - \frac{1}{\tau}(\mu\mathcal{S} + \mathcal{S}^{-1}\overline{\mu}) = M_4 + [(\mathcal{S}\mu + \overline{\mu}\mathcal{S}^{-1}), 1/\tau].$$

But M_4 was already shown to be Fredholm, so we conclude that M_6 and thus M_2 is left semi-Fredholm. We have finally proven that M is Fredholm. It is again legitimate to replace μ and ν by $t\mu$ and $t\nu$ for $0 \leq t \leq 1$, which gives a continuous path between M and the identity, so that M has index zero. It remains to prove that the kernel of M is trivial. Let us recall the original definition of M, which can be written as

$$\begin{aligned} M &= 1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}}(1 - \mu\mathcal{S})^{-1}\nu\overline{\mathcal{S}} \\ &= (1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}})(1 + (1 - \mu\mathcal{S})^{-1}\nu\overline{\mathcal{S}}) \\ &= (1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}})(1 - \mu\mathcal{S})^{-1}(1 - \mu\mathcal{S} + \nu\overline{\mathcal{S}}). \end{aligned}$$

Thus if it is shown that $(1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}})\omega = 0$ implies $\omega = 0$, then $M\omega = 0$ implies $\omega = 0$ as well. Assume therefore that

$$\omega = \mu\mathcal{S}\omega + \nu\overline{\mathcal{S}}\omega$$

for some $\omega \in L^p$. From the assumption that μ and ν are in C_0 we see that ω has compact support and hence lies in L^2. Now we have the estimate

$$\int |\omega|^2 = \int |\mu\mathcal{S}\omega + \nu\overline{\mathcal{S}\omega}|^2 \leq \int (|\mu||\mathcal{S}\omega| + |\nu||\overline{\mathcal{S}\omega}|)^2 \leq k^2 \int |\omega|^2,$$

from which we see that $\omega = 0$ as wanted. Thus M has trivial kernel.

We finally get that T and R from before are well-defined, and hence for each $h \in L^p$ we find a solution $\omega \in L^p$ to $(1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}})\omega = h$. Since in the process we actually proved that $1 - \mu\mathcal{S} - \nu\overline{\mathcal{S}}$ has trivial kernel, this operator is now known to be invertible for every $p \geq 2$.

Now to extend this to L^q for $1 < q \leq 2$. Consider again two bounded operators $T', R' : L^p \mapsto L^p$. Then

$$f \mapsto \langle T'f, h\rangle + \langle R'f, \overline{h}\rangle$$

is a continuous linear functional on L^p, so there exists $\omega \in L^q$ such that

$$\langle T'f, h\rangle + \langle R'f, \overline{h}\rangle = \langle f, \omega\rangle \tag{4.9}$$

for all f. We would now like to determine T' and R' so that (4.9) implies that

$$\langle f, (1-\mu\mathcal{S}-\nu\overline{\mathcal{S}})\omega\rangle = \langle f, h\rangle$$

for all $f \in L^p$. We use the symmetry properties of $\mathcal{S}$ repeatedly to calculate that

$$\begin{aligned}
\langle f, (1-\mu\mathcal{S}-\nu\overline{\mathcal{S}})\omega\rangle &= \langle (1-\mathcal{S}\mu)f, \omega\rangle - \langle \nu f, \mathcal{S}^{-1}\overline{\omega}\rangle \\
&= \langle (1-\mathcal{S}\mu)f, \omega\rangle - \langle \mathcal{S}^{-1}\nu f, \overline{\omega}\rangle \\
&= \langle T'(1-\mathcal{S}\mu)f, h\rangle + \langle R'(1-\mathcal{S}\mu)f, \overline{h}\rangle - \overline{\left\langle \overline{\mathcal{S}^{-1}\nu f}, \omega\right\rangle} \\
&= \langle T'(1-\mathcal{S}\mu)f, h\rangle + \langle R'(1-\mathcal{S}\mu)f, \overline{h}\rangle \\
&\quad - \left\langle \overline{T'}\,\overline{\mathcal{S}^{-1}}\nu f, \overline{h}\right\rangle - \left\langle \overline{R'}\,\overline{\mathcal{S}^{-1}}\nu f, h\right\rangle
\end{aligned}$$

So we must have that

$$\begin{cases} R'(1-\mathcal{S}\mu) - \overline{T'}\,\overline{\mathcal{S}^{-1}}\nu &= 0 \\ T'(1-\mathcal{S}\mu) - \overline{R'}\,\overline{\mathcal{S}^{-1}}\nu &= 1 \end{cases}.$$

From the first equation we get $R' = \overline{T'}\,\overline{\mathcal{S}^{-1}}\nu(1-\mathcal{S}\mu)^{-1}$ and plugging this in the second one gives

$$T'(1-\mathcal{S}\mu-\overline{\mathcal{S}^{-1}}\nu(1-\mathcal{S}\mu)^{-1}\overline{\mathcal{S}^{-1}}\nu) = 1.$$

Thus we are again at a question of invertibility. Note the analogy with the basic Beltrami equation: To solve it in L^p we had to invert $1-\mu\mathcal{S}$ and to solve it in L^q we had to invert $1-\mathcal{S}\mu$. We first calculate that

$$1-\mathcal{S}\mu-\overline{\mathcal{S}^{-1}}\nu(1-\mathcal{S}\mu)^{-1}\overline{\mathcal{S}^{-1}}\nu = 1-\mathcal{S}\mu-\mathcal{S}\overline{\nu}(1-\mathcal{S}^{-1}\overline{\mu})^{-1}\mathcal{S}^{-1}\nu. \quad (4.10)$$

As with proving the invertibility of $1-\mathcal{S}\mu$, we conjugate M with $\mathcal{S}$ to get

$$\begin{aligned}
\mathcal{S}M\mathcal{S}^{-1} &= \mathcal{S}(1-\mu\mathcal{S}-\nu\mathcal{S}^{-1}(1-\overline{\mu}\mathcal{S}^{-1})^{-1}\overline{\nu}\mathcal{S})\mathcal{S}^{-1} \\
&= 1-\mathcal{S}\mu-\mathcal{S}\nu\mathcal{S}^{-1}(1-\overline{\mu}\mathcal{S}^{-1})^{-1}\overline{\nu} \\
&= 1-\mathcal{S}\mu-\mathcal{S}\nu(\mathcal{S}-\overline{\mu})^{-1}\overline{\nu} \\
&= 1-\mathcal{S}\mu-\mathcal{S}\nu(1-\mathcal{S}^{-1}\overline{\mu})^{-1}\mathcal{S}^{-1}\overline{\nu}.
\end{aligned}$$

Now we are done, since the last expression only differs from (4.10) by a replacement of ν by $\overline{\nu}$. Thus even our generalized equation is solvable in L^p for $1 < p < \infty$, as long as there is sufficient global regularity for μ and ν. □

4.2 Ellipticity

We now briefly turn our attention to the concept of ellipticity that previously manifested itself as the conditions $||\mu||_\infty < 1$ and $||\,|\mu| + |\nu|\,||_\infty < 1$. For a more ample discussion of the subject see [AIM09], Chapter 7. Ellipticity itself is a property that is satisfied by specific differential operators

$$\mathcal{L} = \sum_{|\alpha| \leq m} A_\alpha \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}}$$

acting on functions of k real variables valued in a space $\mathbb{E}$ which for us will be $\mathbb{R}^n$ or $\mathbb{C}^n$. The coefficients A_α are at the moment constant real or complex matrices. Such an operator is said to be elliptic if the so-called principal symbol

$$\mathcal{P}(\zeta) = \sum_{|\alpha|=m} \zeta^\alpha A_\alpha : \mathbb{E} \mapsto \mathbb{E}$$

is invertible for all $\zeta \in \mathbb{R}^k \setminus \{0\}$. In the case where the matrices A_α are allowed to depend on the variables $x_1, \ldots, x_k$ one has to be more careful. We say that two elliptic operators

$$\mathcal{L}_0 = \sum_{|\alpha| \leq m} A_\alpha^0 \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}} \text{ and } \mathcal{L}_1 = \sum_{|\alpha| \leq m} A_\alpha^0 \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}}$$

are homotopic if for all α there exist continuous maps A_α from $[0,1]$ to either the space of real or complex matrices such that $A_\alpha(0) = A_\alpha^0$, $A_\alpha(1) = A_\alpha^1$ and the operator

$$\mathcal{L}_t = \sum_{|\alpha| \leq m} A_\alpha(t) \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}}$$

is elliptic for all t. Now we define that an operator with variable coefficients

$$\mathcal{L} = \sum_{|\alpha| \leq m} A_\alpha(x_1, \ldots, x_k) \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}}$$

is elliptic if the operators with constant coefficients

$$\mathcal{L}_y = \sum_{|\alpha| \leq m} A_\alpha(x_1^0, \cdots, x_k^0) \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}}$$

are elliptic and lie in the same homotopy class for almost every point $x = (x_1^0, \cdots, x_k^0) \in \mathbb{R}^k$.

Let us first actually verify that this definition of ellipticity does not contradict the examples seen before. We prove a general fact: if $\mathcal{M}$ and $\mathcal{N}$

are complex n-by-n matrices depending on some complex parameter z that satisfy the almost everywhere bound $||\mathcal{M}(z)|| + ||\mathcal{N}(z)|| \leq k < 1$ (here $||\cdot||$ denotes the operator norm) then the operator

$$\mathcal{L} = \partial_{\overline{z}} - \mathcal{M}\partial_z - \mathcal{N}\overline{\partial_z}$$

acting on functions from $\mathbb{C}$ to $\mathbb{C}^n$ is elliptic. Going back to real coordinates $\mathcal{L}$ may be written as

$$\mathcal{L} = \frac{1}{2}(1 - \mathcal{M} - \mathcal{N})\partial_x + \frac{i}{2}(1 + \mathcal{M} - \mathcal{N})\partial_y,$$

so we vill have to verify that the principal symbol

$$\begin{aligned}\mathcal{P}_z(\zeta_1, \zeta_2) &= \frac{1}{2}\zeta_1 \left[1 - \mathcal{M}(z) - \mathcal{N}(z)\right] + \frac{i}{2}\zeta_2 \left[1 + \mathcal{M}(z) - \mathcal{N}(z)\right] \\ &= \frac{1}{2}\left[\zeta - \overline{\zeta}\mathcal{M}(z) - \zeta\mathcal{N}(z)\right]\end{aligned}$$

is invertible for all $\zeta = \zeta_1 + i\zeta_2 \neq 0$ and almost every z. This follows from the estimate

$$\begin{aligned}||\mathcal{P}(\zeta_1, \zeta_2)v|| &\geq \frac{1}{2}(|\zeta||v| - |\zeta|\,||\mathcal{M}(z)|||v| - |\zeta|\,||\mathcal{N}(z)||\,|v|) \\ &\geq \frac{|\zeta|}{2}(1 - ||\mathcal{M}(z)|| - ||\mathcal{N}(z)||)|v| \\ &\geq \frac{|\zeta|}{2}(1 - k)|v|.\end{aligned}$$

Now it remains to note that the operators

$$\mathcal{L}_0 = \frac{1}{2}(1 - \mathcal{M}(z_0) - \mathcal{N}(z_0))\partial_x + \frac{i}{2}(1 + \mathcal{M}(z_0) - \mathcal{N}(z_0))\partial_y$$

and

$$\mathcal{L}_1 = \frac{1}{2}(1 - \mathcal{M}(z_1) - \mathcal{N}(z_1))\partial_x + \frac{i}{2}(1 + \mathcal{M}(z_1) - \mathcal{N}(z_1))\partial_y$$

are homotopic via the elliptic operators $\mathcal{L}_t = t\mathcal{L}_0 + (1 - t)\mathcal{L}_1$, which shows that $\mathcal{L}$ is elliptic as claimed.

Remark. The assumption $||\mathcal{M}(z)|| + ||\mathcal{N}(z)|| \leq k < 1$ could have been replaced with the weaker condition that $||\mathcal{M}(z)|| + ||\mathcal{N}(z)|| < 1$ holds for almost every z. It is customary to call the case where the stronger condition holds *uniform ellipticity* and the case when only the weaker one holds *degenerate ellipticity*. As previously seen the uniform bounds have been crucial in the proofs of the invertibility of certain operators, which is why we will mainly focus on uniformly elliptic operators in the future.

4.3 Higher dimensions

We now seek to extend the previous theorems to a higher-dimensional setting. What we increase in dimension is, however, not the domain but the range of our functions: We consider maps

$$f = (f_1, \dots, f_n) : \mathbb{C} \mapsto \mathbb{C}^n$$

whose components are measurable and whose L^p-norm defined by

$$||f||_p = \left(\int_{\mathbb{C}} |f(z)|^p dz \right)^{\frac{1}{p}} \tag{4.11}$$

is finite. This space is denoted by $L^p(\mathbb{C}, \mathbb{C}^n)$ as discussed in the appendix. We may then identify $L^p(\mathbb{C}, \mathbb{C}^n)$ as the direct sum of n complex L^p-spaces, since the taxicab norm

$$||f||_{p,\text{taxi}} = \sum_{k=1}^{n} ||f_k||_p$$

defined in the usual direct sum construction is easily seen to be equivalent with the norm defined in (4.11) by a few elementary inequalities. Thus in particular $L^p(\mathbb{C}, \mathbb{C}^n)$ is a Banach space.

Let $\mathcal{M} = (\mu_{ij})$ and $\mathcal{N} = (\nu_{ij})$ denote two n-by-n matrices whose coefficients are complex-valued C_0-functions. We also assume a form of uniform ellipticity: If $|| \cdot ||$ denotes the matrix operator norm, define

$$||M||_\infty = \operatorname{ess\,sup}_{z \in \mathbb{C}} ||M(z)||$$

for any n-by-n matrix M of measurable coefficients. Then our standing assumption is that $||\mathcal{M}||_\infty + ||\mathcal{N}||_\infty < 1$. Our aim is to solve the equation

$$f_{\bar{z}} - \mathcal{M} f_z = h \tag{4.12}$$

and its generalized version

$$f_{\bar{z}} - \mathcal{M} f_z - \mathcal{N} \overline{f_z} = h, \tag{4.13}$$

where the complex derivatives are defined componentwise. Note that the second equation again covers the case $f_{\bar{z}} = H(z, f_z)$, where H is linear in the second component. Our ellipticity assumption, however, is much stronger than requiring an uniform bound on the operator norm of $H(z, \cdot)$.

Observe that if we also define an analogue of the Beurling transform componentwise as $\mathcal{S}f = (\mathcal{S}f_1, \dots, \mathcal{S}f_n)$ then these two equations may again be reduced to the invertibility of the operators

$$1 - \mathcal{M}\mathcal{S} \quad \text{and} \quad 1 - \mathcal{M}\mathcal{S} - \mathcal{N}\overline{\mathcal{S}},$$

where the second one is really just a real-linear map instead of a $\mathbb{C}$-linear operator on $L^p(\mathbb{C}, \mathbb{C}^n)$. The following theorem will be proven

Theorem 4.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be as before. Then for any $1 < p < \infty$ the operators*

$$(i) \quad 1 - \mathcal{M}\mathcal{S}$$
$$(ii) \quad 1 - \mathcal{M}\mathcal{S} - \mathcal{N}\overline{\mathcal{S}}$$

are invertible on $L^p(\mathbb{C}, \mathbb{C}^n)$.

We will have to verify first that n-dimensional versions of the auxiliary results we used in the previous subsection still hold.

To start off let us concern ourselves with the boundedness of our operators in question. Note the estimate

$$||Mf||_p = \left(\int_{\mathbb{C}} |M(z)f(z)|^p dz \right)^{\frac{1}{p}} \leq ||M||_\infty ||f||_p$$

for $f \in L^p(\mathbb{C}, \mathbb{C}^n)$ and M a n-by-n matrix of measurable coefficients. This shows that each matrix bounded in the norm $|| \cdot ||_\infty$ naturally defines a bounded operator on $L^p(\mathbb{C}, \mathbb{C}^n)$ as expected.

Given any bounded operator T on L^p and its componentwise defined analogue $\mathcal{T}$ on $L^p(\mathbb{C}, \mathbb{C}^n)$, we easily find the estimate $||\mathcal{T}f||_{p,\text{taxi}} \leq ||T||_p ||f||_{p,\text{taxi}}$, which by the equivalence of norms implies the estimate

$$||\mathcal{T}f||_p \leq C_{p,n} ||T||_p ||f||_p,$$

for some constant $C_{p,n}$ depending only on p and n. Especially this shows that Theorem 4.4 carries over to the higher-dimensional setting:

$$||\mathcal{S}^m||_p \leq B_{p,n}(1 + m^2) \tag{4.14}$$

for any integer m and $B_{p,n}$ only depending on $p \in (1, \infty)$ and $n \in \mathbb{Z}_+$. Note that the identities $\mathcal{S}^{-1}f = \overline{\mathcal{S}\overline{f}}$ and $||\mathcal{S}f||_2 = ||f||_2$ also hold for our componentwise definition. We hope that the fact that we denote by $\mathcal{S}$ both the n-dimensional and the basic version of Beurling transform causes no confusion.

The compactness of certain commutators played an important role in the solving of the basic and generalized Beltrami equations before. We use a sequential argument to show that there is no trouble generalizing this for commutators of $\mathcal{S}$ with matrices of VMO-coefficients. Given a function f, note that for $M = (m_{ij})$ a n-by-n matrix we have that

$$[\mathcal{S}, M]f = \mathcal{S}Mf - M\mathcal{S}f = (g_1, \ldots, g_n),$$

where $g_k = \sum_{l=1}^n [\mathcal{S}, m_{kl}] f_l$. Now given a sequence $(f^{(N)}) \subset L^p(\mathbb{C}, \mathbb{C}^n)$ with norms $||f^{(N)}||_p \leq 1$ for all N, it suffices to verify that the sequence

$([\mathcal{S}, M]f^{(N)})$ has a subsequence converging in $L^p(\mathbb{C}, \mathbb{C}^n)$. Note that if the coefficients m_{ij} are assumed to be in VMO, then by Theorem 4.2 and the estimate $||f_l||_p \leq ||f||_p \leq 1$ each sequence $([\mathcal{S}, m_{kl}]f_l^{(N)})$ has a convergent subsequence in L^p. By passing to a subsequence sufficiently many times we may assume that each of these sequences converges in L^p. But then it is clear that the sequence $([\mathcal{S}, M]f^{(N)})$ can be written as a finite sum of convergent sequences in $L^p(\mathbb{C}, \mathbb{C}^n)$, so that it converges itself and we are done.

Since Fredholm theory also applies to any Banach space, we are now ready to start solving equations (4.12) and (4.13). The proof of the invertibility of $1 - \mathcal{MS}$ is similar enough to the one-dimensional case that we will only give a sketch of the main points:

Let $p \geq 2$. To prove that $1 - \mathcal{MS}$ is Fredholm on $L^p(\mathbb{C}, \mathbb{C}^n)$, we again write

$$(1 - \mathcal{MS}) \sum_{k=1}^{N-1} (\mathcal{MS})^k = 1 - \mathcal{M}^N \mathcal{S}^N + K,$$

where K is a sum of multiples of $[\mathcal{S}, \mathcal{M}]$ and thus compact. The estimate $||\mathcal{M}^N \mathcal{S}^N||_p \leq ||\mathcal{M}||_p^N B_{p,n}(1 + N^2)$ again shows that $1 - \mathcal{M}^N \mathcal{S}^N$ is invertible for some N and thus our operator is Fredholm. Replacing $\mathcal{M}$ by $t\mathcal{M}$ for $t \in [0, 1]$ yields that the index is zero, and if ω is to solve

$$\omega = \mathcal{MS}\omega,$$

then the compactness of the support of each coefficient of $\mathcal{M}$ implies the same for each component of ω. Thus ω is in $L^2(\mathbb{C}, \mathbb{C}^n)$, and the estimate $||\omega||_2 \leq ||\mathcal{M}||_\infty ||\omega||_2$ shows that $\omega = 0$, i.e. the kernel of $1 - \mathcal{MS}$ is trivial. We conclude that $1 - \mathcal{MS}$ is invertible, and so is $1 - \mathcal{SM} = \mathcal{S}(1 - \mathcal{MS})\mathcal{S}^{-1}$.

One might have doubts whether the dual of $L^p(\mathbb{C}, \mathbb{C}^n)$ is $L^q(\mathbb{C}, \mathbb{C}^n)$, but these are quickly diminished by the following basic result

Theorem 4.8. *Let $B_1, \ldots, B_n$ be topological vector spaces over the same scalar field $\mathbb{K}$, and let $B_1^*, \ldots, B_n^*$ denote their respective duals. Define also $B = \bigoplus_{k=1}^n B_k$. Then it holds that $B^* \cong \bigoplus_{k=1}^n B_k^*$.*

Proof. Assume that we are given a linear functional $\mathcal{L} : B \mapsto \mathbb{K}$. We may then construct continuous linear functionals $\mathcal{L}_k : B_k \mapsto \mathbb{C}$ by mapping

$$f \mapsto F_k(f) \mapsto \mathcal{L}F_k(f),$$

where $F_k(f)$ has k:th component f and other components zero. If now $b = (b_1, \ldots, b_n) \in B$ is arbitrary then

$$\mathcal{L}b = \mathcal{L} \sum_{k=1}^n F_k(b_k) = \sum_{k=1}^n \mathcal{L}_k b_k.$$

Now it is easy to see that the map $B^* \mapsto \bigoplus_{k=1}^n B_k^*$ given by $\mathcal{L} \mapsto (\mathcal{L}_1, \ldots, \mathcal{L}_k)$ is an isomorphism, and thus we have our desired result. $\square$

From the previous theorem it follows that the duality $(L^p(\mathbb{C}, \mathbb{C}^n))^* = L^q(\mathbb{C}, \mathbb{C}^n)$ holds, and we note that the same proof of extending the invertibility to $1 < q \leq 2$ works even in the higher dimensional setting due to its purely algebraic nature. We are now done with (i) in Theorem 4.7.

We now invert the real-linear map (ii) in Theorem 4.7. The trick is to simply think of $L^p(\mathbb{C}, \mathbb{C}^n)$ as a real Banach space instead of a complex one, so that the familiar results of Fredholm theory apply. Note that commutators of $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{S}$ are compact, but for $\overline{\mathcal{S}}$ this might not be the case. Luckily the problem is just a matter of conjugation, since we have the identity

$$\overline{M}\,\overline{\mathcal{S}}f - \overline{\mathcal{S}}Mf = \overline{M\mathcal{S}f - \mathcal{S}Mf}$$

and hence the operator $\overline{M}\,\overline{\mathcal{S}} - \overline{\mathcal{S}}M$ is compact for any M of VMO-coefficients.

We again first assume that $p \geq 2$. Denote for convenience $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_1 = \mathcal{N}$, $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{S}_1 = \overline{\mathcal{S}}$. If we also define by $Q_m = \sum_{k=0}^{m-1}(\mathcal{M}_0\mathcal{S}_0 + \mathcal{M}_1\mathcal{S}_1)^k$ the partial sums of the Neumann series for $\mathcal{M}_0\mathcal{S}_0 + \mathcal{M}_1\mathcal{S}_1$ we find that

$$\begin{aligned} Q_m(1 - \mathcal{M}_0\mathcal{S}_0 - \mathcal{M}_1\mathcal{S}_1) &= 1 - (\mathcal{M}_0\mathcal{S}_0 + \mathcal{M}_1\mathcal{S}_1)^m \\ &= 1 - \sum_{\alpha \in \{0,1\}^m} \mathcal{M}_{\alpha_1}\mathcal{S}_{\alpha_1} \cdots \mathcal{M}_{\alpha_m}\mathcal{S}_{\alpha_m} \\ &= 1 - \sum_{\alpha \in \{0,1\}^m} \mathcal{M}_{\alpha_1}^{\beta_{\alpha,1}} \cdots \mathcal{M}_{\alpha_m}^{\beta_{\alpha,m}} \mathcal{S}_{\alpha_1} \cdots \mathcal{S}_{\alpha_m} + K, \end{aligned}$$

where K is compact and we have used the notation $\mathcal{M}_{\alpha_k}^{\beta_{\alpha,k}}$ to denote either $\mathcal{M}_{\alpha_k}$ or $\overline{\mathcal{M}_{\alpha_k}}$, depending on how many times we have to swap $\mathcal{S}_1\mathcal{M}_{\alpha_k}$ with $\overline{\mathcal{M}_{\alpha_k}}\mathcal{S}_1$.

To show that $1 - \mathcal{M}_0\mathcal{S}_0 - \mathcal{M}_1\mathcal{S}_1$ is Fredholm it again suffices to show that the p-norm of the second term on the utmost right hand side above decays to zero as $m \to \infty$. We therefore seek to estimate the norms of the products $\mathcal{S}_{\alpha_1} \cdots \mathcal{S}_{\alpha_m}$, or equivalently the norm of a product of m instances of $\mathcal{S}$ and $\overline{\mathcal{S}}$. From the identity $\mathcal{S}\overline{f} = \overline{\mathcal{S}^{-1}f}$ it follows inductively that

$$\begin{aligned} ||\mathcal{S}^k\overline{\mathcal{S}}f||_p &= ||\mathcal{S}^{k-1}\overline{\mathcal{S}^{-1}\mathcal{S}f}||_p = ||\mathcal{S}^{k-2}\overline{\mathcal{S}^{-2}\mathcal{S}f}||_p \\ &= \cdots = ||\overline{\mathcal{S}^{-k}\mathcal{S}f}||_p = ||\mathcal{S}^{1-k}f||_p \end{aligned}$$

for any integer k. This shows that the norm of any product of the operators $\mathcal{S}$ and $\overline{\mathcal{S}}$ may be unraveled to produce the norm of a power of $\mathcal{S}$, for example

$$||\mathcal{S}\mathcal{S}\overline{\mathcal{S}}\overline{\mathcal{S}}\mathcal{S}\mathcal{S}\overline{\mathcal{S}}||_p = ||\mathcal{S}^{-1}\overline{\mathcal{S}}\mathcal{S}\overline{\mathcal{S}}||_p = ||\mathcal{S}^2\mathcal{S}\overline{\mathcal{S}}||_p = ||\mathcal{S}^{-2}||_p.$$

And it is clear that the resulting exponent is always at most the number of original factors, so in view of Theorem 4.4 this results in the estimate

$$||\mathcal{S}_{\alpha_1} \cdots \mathcal{S}_{\alpha_m}||_p \leq B_{p,n}(1+m^2).$$

We now see that

$$\begin{aligned}
&\left|\left|\sum_{\alpha\in\{0,1\}^m} \mathcal{M}_{\alpha_1}^{\beta_{\alpha,1}} \cdots \mathcal{M}_{\alpha_m}^{\beta_{\alpha,m}} \mathcal{S}_{\alpha_1} \cdots \mathcal{S}_{\alpha_m}\right|\right|_p \\
&\leq B_{p,n}(1+m^2) \sum_{\alpha\in\{0,1\}^m} ||\mathcal{M}_{\alpha_1}^{\beta_{\alpha,1}} \cdots \mathcal{M}_{\alpha_m}^{\beta_{\alpha,m}}||_\infty \\
&\leq B_{p,n}(1+m^2) \sum_{\alpha\in\{0,1\}^m} ||\mathcal{M}_{\alpha_1}||_\infty \cdots ||\mathcal{M}_{\alpha_m}||_\infty \\
&= B_{p,n}(1+m^2)(||\mathcal{M}_0||_\infty + ||\mathcal{M}_1||_\infty)^m,
\end{aligned}$$

which indeed tends to zero as $m \to \infty$. The result is that $1 - \mathcal{M}\mathcal{S} - \mathcal{N}\overline{\mathcal{S}}$ is Fredholm. We can also prove in the same way as before that the index and kernel are both zero, which shows that $1 - \mathcal{M}\mathcal{S} - \mathcal{N}\overline{\mathcal{S}}$ is indeed invertible on L^p for $p \geq 2$.

Let then $1 < q \leq 2$. We aim to do the same type of duality argument as before, but one should be careful to specify what we actually mean by the dual in this case. As $L^p(\mathbb{C}, \mathbb{C}^n)$ is now considered a real Banach space, it is most natural to define the dual $(L^p(\mathbb{C}, \mathbb{C}^n))^{*,\mathbb{R}}$ as the real vector space of continuous linear functionals $\mathcal{L} : L^p(\mathbb{C}, \mathbb{C}^n) \mapsto \mathbb{R}$. Replacing complex variables by real ones we see that $L^p(\mathbb{C}, \mathbb{C}^n)$ considered as a real Banach space is isomorphic to $L^p(\mathbb{R}^2, \mathbb{R}^{2n}) = (L^p(\mathbb{R}^2, \mathbb{R}))^{2n}$. Theorem 4.8 now implies that

$$(L^p(\mathbb{C}, \mathbb{C}^n))^{*,\mathbb{R}} = L^q(\mathbb{R}^2, \mathbb{R})^{2n} = L^q(\mathbb{R}^2, \mathbb{R}^{2n}).$$

Thus each continuous linear functional $\mathcal{L} : L^p(\mathbb{C}, \mathbb{C}^n) \mapsto \mathbb{R}$ is uniquely represented by a function $g \in L^q(\mathbb{C}, \mathbb{C}^n)$ with the duality pairing

$$\begin{aligned}
\mathcal{L}f &= \langle f, g\rangle_{\mathbb{R}} \\
&= \sum_{k=1}^n \left(\int \operatorname{Re} f_k \operatorname{Re} g_k + \int \operatorname{Im} f_k \operatorname{Im} g_k \right) \\
&= \sum_{k=1}^n \int \operatorname{Re}(f_k \overline{g_k}) \\
&= \operatorname{Re} \int \sum_{k=1}^n f_k \overline{g_k} \\
&= \operatorname{Re} \langle f, \overline{g} \rangle .
\end{aligned}$$

Let then $h \in L^q(\mathbb{C}, \mathbb{C}^n)$ be given, and we will look for $\omega \in L^q(\mathbb{C}, \mathbb{C}^n)$ such that

$$\left\langle f, (1 - \mathcal{M}\mathcal{S} - \mathcal{N}\overline{\mathcal{S}})\omega \right\rangle_{\mathbb{R}} = \langle f, h \rangle_{\mathbb{R}}$$

for all $f \in L^p(\mathbb{C}, \mathbb{C}^n)$. Denote by $\mathcal{C} : L^p(\mathbb{C}, \mathbb{C}^n) \mapsto L^p(\mathbb{C}, \mathbb{C}^n)$ the componentwise complex conjugation map, especially $\overline{\mathcal{S}} = \mathcal{C}\mathcal{S}$. We define a linear functional $\mathcal{L}$ in $(L^p(\mathbb{C}, \mathbb{C}^n))^{*,\mathbb{R}}$ by

$$\mathcal{L}f = \left\langle (1 - \mathcal{C}\mathcal{S}\mathcal{M}^T\mathcal{C} - \mathcal{C}\mathcal{S}\overline{\mathcal{N}}^T)^{-1} f, h \right\rangle_{\mathbb{R}},$$

where the $L^p(\mathbb{C}, \mathbb{C}^n)$ invertibility of the operator

$$1 - \mathcal{C}\mathcal{S}\mathcal{M}^T\mathcal{C} - \mathcal{C}\mathcal{S}\overline{\mathcal{N}}^T$$

is due to the identity

$$1 - \mathcal{C}\mathcal{S}\mathcal{M}^T\mathcal{C} - \mathcal{C}\mathcal{S}\overline{\mathcal{N}}^T = \mathcal{C}\mathcal{S}(1 - \mathcal{M}^T\mathcal{S} - \overline{\mathcal{N}}^T\overline{\mathcal{S}})\mathcal{S}^{-1}\mathcal{C}.$$

Now there exists a function $\omega \in L^q(\mathbb{C}, \mathbb{C}^n)$ such that $\mathcal{L}f = \langle f, \omega \rangle_{\mathbb{R}}$ for all f. Especially we find that

$$\begin{aligned}
\left\langle f, (1 - \mathcal{M}\mathcal{S} - \mathcal{N}\overline{\mathcal{S}})\omega \right\rangle_{\mathbb{R}} &= \langle f, \omega \rangle_{\mathbb{R}} - \operatorname{Re}\left\langle f, \overline{\mathcal{M}}\,\overline{\mathcal{S}}\omega \right\rangle - \operatorname{Re}\left\langle f, \overline{\mathcal{N}}\mathcal{S}\omega \right\rangle \\
&= \langle f, \omega \rangle_{\mathbb{R}} - \operatorname{Re}\left\langle \mathcal{M}^T\overline{f}, \mathcal{S}\omega \right\rangle - \operatorname{Re}\left\langle \overline{\mathcal{N}}^T f, \mathcal{S}\omega \right\rangle \\
&= \langle f, \omega \rangle_{\mathbb{R}} - \operatorname{Re}\left\langle \overline{\mathcal{S}}\mathcal{M}^T\overline{f}, \overline{\omega} \right\rangle - \operatorname{Re}\left\langle \overline{\mathcal{S}}\,\overline{\mathcal{N}}^T f, \overline{\omega} \right\rangle \\
&= \left\langle f - \overline{\mathcal{S}}\mathcal{M}^T\overline{f} - \overline{\mathcal{S}}\,\overline{\mathcal{N}}^T f, \omega \right\rangle_{\mathbb{R}} \\
&= \left\langle (1 - \mathcal{C}\mathcal{S}\mathcal{M}^T\mathcal{C} - \mathcal{C}\mathcal{S}\overline{\mathcal{N}}^T) f, \omega \right\rangle_{\mathbb{R}} \\
&= \langle f, h \rangle_{\mathbb{R}}
\end{aligned}$$

as wanted. Thus Theorem 4.7 is proven. □

5 Quaternions

Let us now generalize the theory of the Beltrami equation in a different direction. We will concern ourselves with the four-dimensional algebra $\mathbb{H}$ over the reals, called the quaternions. This algebra is just one property away from being a field as the product happens to be noncommutative. The basis elements, usually denoted by $1, i, j$, and k, will be denoted by e_1, e_2, e_3 and e_4 for convenience. The following multiplication table fixes the product in $\mathbb{H}$.

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	$-e_1$	e_2
e_4	e_4	e_3	$-e_2$	$-e_1$

where each entry stands for the product of the corresponding elements of the first row and first column, in that order. The usual identification of $\mathbb{H}$ with $\mathbb{R}^4$ also lets us define the usual Euclidian norm $|\cdot|$ on $\mathbb{H}$. Defining for a quaternion $h = h_1e_1+h_2e_2+h_3e_3+h_4e_3$ its conjugate $\overline{h} = h_1e_1-h_2e_2-h_3e_3-h_4e_4$ we find the nice property $h\overline{h} = \overline{h}h = |h|^2$.

We introduce some additional notation for convenience. Given two ordered quadruplets $a, b \in \mathbb{H}^4$ of quaternions we define the componentwise product by

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4.$$

Let us make some additional notes about this convention.

- The standard basis can be thought of as the quadruplet $e = (e_1, e_2, e_3, e_4)$.
- We associate with each quaternion $h = \sum_{m=1}^4 h_m e_m$ the quadruplet $Q_h = (h_1, h_2, h_3, h_4)$ so that $h = Q_h \cdot e$.
- Conjugation of a quadruplet $a \in \mathbb{H}^4$ is defined componentwise, i.e. $\overline{a} = (\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4})$. Note that $\overline{a \cdot b} = \overline{b} \cdot \overline{a}$.

Let now a map $T : \mathcal{B}_1 \mapsto \mathcal{B}_2$ be given, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are some classes of functions from $\mathbb{R}^4$ to $\mathbb{C}$. We may generalize the map T to functions $f : \mathbb{H} \mapsto \mathbb{H}$ whose components are in $\mathcal{B}_1$ by defining

$$Tf = T(f^1e_1+f^2e_2+f^3e_3+f^4e_4) = Tf^1e_1+Tf^2e_2+Tf^3e_3+Tf^4e_4. \quad (5.1)$$

This definition enjoys the nice property that

$$T(F \cdot \phi) = TF \cdot \phi$$

for any $\phi \in \mathbb{H}^4$ and $F : \mathbb{H} \mapsto \mathbb{H}^4$, where TF is defined componentwise.

Remark 5.1. Note that we have made a choice in the above definition: Due to the lack of commutativity we could as well have defined $Tf = \sum_{m=1}^4 e_m Tf^m$. This notational choice will reflect itself virtually everywhere, which is why the theory splits into 'left'- and 'right' theories depending on which side one writes the basis elements in (5.1) respectively. Although we will work with the 'right' theory, the 'left' theory is not wrong either and is preferred in most of the papers on the subject [Käh00]. Nevertheless, if one writes each following quaternion product in the reverse order we obtain results for the 'left'-theory and vice versa.

In our quest to generalize the Beltrami equations to the quaternionic setting we must first find analogues of the Cauchy-Riemann complex derivatives ∂_z and $\partial_{\overline{z}}$. However, due to the sheer amount of viable choices of differential operators we first look at the situation in all generality. Given $\phi \in \mathbb{H}^4$ we define its associated differential operator ∂_ϕ by

$$\partial_\phi f = \nabla f \cdot \phi = \frac{\partial f}{\partial h_1}\phi_1 + \frac{\partial f}{\partial h_2}\phi_2 + \frac{\partial f}{\partial h_3}\phi_3 + \frac{\partial f}{\partial h_4}\phi_4. \tag{5.2}$$

Unfortunately, this differential operator fails to satisfy the Leibniz rule since $\partial_\phi(fg) = \sum_{k=1}^4 (\partial f/\partial h_k) g\phi_k + f\partial_\phi g$. As in the complex case, we can use these operators to obtain factorizations of the four-dimensional Laplacian as $2\Delta = \partial_\phi\partial_\psi = \partial_\psi\partial_\phi$ for any ϕ, ψ satisfying the orthogonalty condition $\phi_i\psi_j + \phi_j\psi_i = 2\delta_{ij}$ for all i, j.

A natural follow-up question would of course be to ask if we could also generalize the Beurling transform $\mathcal{S}$ to obtain operators $\mathcal{S}_{\phi\psi}$ for which the identity $\mathcal{S}_{\phi\psi}\partial_\phi f = \partial_\psi f$ would be valid for sufficiently nice functions f. We look for a solution by considering the associated Fourier multipliers, where the Fourier transform is generalized via Definition 5.1 (or equivalently $\hat{f}(\xi) = \int_{\mathbb{H}} e^{-2\pi i(\xi\cdot h)} f(h)dh$). When speaking of Fourier multipliers, however, one must be cautious due to the fact that we are in a noncommutative setting. As the Fourier transform $\widehat{f}$ of f is quaternion-valued we will have to make a distinction between factors on the left and right sides of $\widehat{f}$, leading to left-multiplier operators and right-multiplier operators respectively. An operator which has both a left- and right-multiplier is also plausible. An example of this is seen by calculating the multiplier of the differential operator ∂_ϕ as

$$\widehat{\partial_\phi f}(\xi) = (\nabla f \cdot \phi)^\wedge(\xi) = \left(\widehat{\frac{\partial f}{\partial h_1}}, \widehat{\frac{\partial f}{\partial h_2}}, \widehat{\frac{\partial f}{\partial h_3}}, \widehat{\frac{\partial f}{\partial h_4}}\right) \cdot \phi = 2\pi i \widehat{f}(\xi)(Q_\xi \cdot \phi). \tag{5.3}$$

This shows that $\mathcal{S}_{\phi\psi}$ is to be given by the equation

$$\widehat{\mathcal{S}_{\phi\psi} f}(\xi) = \widehat{f}(\xi)(Q_\xi \cdot \phi)^{-1}(Q_\xi \cdot \phi) = \widehat{f}(\xi)\frac{(Q_\xi \cdot \overline{\phi})}{|Q_\xi \cdot \phi|^2}(Q_\xi \cdot \psi). \tag{5.4}$$

To achieve some form of symmetry we make the following standing assumption about ϕ (and ψ):

$$|Q_\xi \cdot \phi| = |\xi| \quad \text{for all } \xi \in \mathbb{H}. \tag{5.5}$$

This holds for example if $\phi_m = \pm e_m$, and is satisfied precisely when the components of ϕ satisfy the orthogonality condition $\mathrm{Re}(\phi_m \overline{\phi_l}) = \delta_{ml}$ for all m and l. Since equation (5.4) now becomes equivalent with

$$\widehat{\mathcal{S}_{\phi\psi} f}(\xi) = \widehat{f}(\xi)\frac{Q_\xi \cdot \overline{\phi}}{|\xi|}\frac{Q_\xi \cdot \psi}{|\xi|}$$

we are invited to look for an operator with right-multiplier $(Q_\xi \cdot \phi)/|\xi|$ for given ϕ. An immediate solution is provided by the four-dimensional Riesz transforms R_m, $m = 1, 2, 3, 4$, with left-multipliers $-i\xi_m/|\xi|$. Given the four-dimensional vector-valued Riesz transform $\mathcal{R} = (R_1, R_2, R_3, R_4)$ and $\phi \in \mathbb{H}^4$ we define the operator $\mathcal{R} \cdot \phi$ by

$$(\mathcal{R} \cdot \phi)f = (\mathcal{R}f) \cdot \phi.$$

Its Fourier multiplier is then given by

$$((\mathcal{R} \cdot \phi)f)^\wedge(\xi) = \widehat{\mathcal{R}f(\xi)} \cdot \phi = -i\widehat{f}(\xi)\left(\frac{\xi_1}{|\xi|}, \frac{\xi_2}{|\xi|}, \frac{\xi_3}{|\xi|}, \frac{\xi_4}{|\xi|}\right) \cdot \phi = -i\widehat{f}(\xi)\frac{Q_\xi \cdot \phi}{|\xi|}.$$

We claim that $\mathcal{S}_{\phi\psi} = -(\mathcal{R} \cdot \psi)(\mathcal{R} \cdot \overline{\phi})$ now gives the desired operator. Indeed,

$$\begin{aligned}
(\mathcal{S}_{\phi\psi}\partial_\phi f)^\wedge(\xi) &= -((\mathcal{R} \cdot \psi)(\mathcal{R} \cdot \overline{\phi})\partial_\phi f)^\wedge(\xi) \\
&= i((\mathcal{R} \cdot \overline{\phi})\partial_\phi f)^\wedge(\xi)\frac{Q_\xi \cdot \psi}{|\xi|} \\
&= \widehat{\partial_\phi f}(\xi)\frac{Q_\xi \cdot \overline{\phi}}{|\xi|}\frac{Q_\xi \cdot \psi}{|\xi|} \\
&= 2\pi i\widehat{f}(\xi)(Q_\xi \cdot \phi)(Q_\xi \cdot \phi)^{-1}(Q_\xi \cdot \psi) \\
&= \widehat{\partial_\psi f}(\xi).
\end{aligned}$$

Thus the identity

$$\mathcal{S}_{\phi\psi}\partial_\phi f = \partial_\psi f$$

becomes valid for $f \in C_0^\infty$. As the Riesz transforms are known to be bounded from $L^p(\mathbb{R}^4, \mathbb{C})$ to itself for $1 < p < \infty$, we also get for free that our operator $\mathcal{S}_{\phi\psi} : L^p(\mathbb{H}, \mathbb{H}) \mapsto L^p(\mathbb{H}, \mathbb{H})$ is bounded.

5.1 L^p-theory of the integral operators

In this subsection we will encounter a few convolution-type integral operators. But before we begin with the theory it should be noted that the usual convolution formula for the Fourier transform fails in the quaternionic setting. If we are given any $f, g : \mathbb{H} \mapsto \mathbb{H}$, then we may write $f = a + be_3$ and $g = c + de_3$, where a, b, c and d are complex-valued. It follows that

$$\begin{aligned}
(f * g)(h) &= \int_{\mathbb{H}} f(\omega)g(h - \omega)d\omega \\
&= \int_{\mathbb{H}} (a(\omega)c(h - \omega) + a(\omega)d(h - \omega)e_3 \\
&\quad + b(\omega)e_3c(h - \omega) + b(\omega)e_3d(h - \omega)e_3)d\omega \\
&= \int_{\mathbb{H}} a(\omega)c(h - \omega)d\omega + \int_{\mathbb{H}} a(\omega)d(h - \omega)e_3d\omega \\
&\quad + \int_{\mathbb{H}} b(\omega)\overline{c(h - \omega)}e_3d\omega - \int_{\mathbb{H}} b(\omega)\overline{d(h - \omega)}d\omega \\
&= (a * c)(h) + (a * d)(h)e_3 + (b * \overline{c})(h)e_3 - (b * \overline{d})(h).
\end{aligned}$$

If we use the notation $\tilde{F}(h) = F(-h)$, then

$$\widehat{f * g} = \widehat{a}\widehat{c} + \widehat{a}\widehat{d}e_3 + \widehat{b}\widehat{\tilde{c}}e_3 - \widehat{b}\widehat{\tilde{d}} = \widehat{a}(\widehat{c} + \widehat{d}e_3) + \widehat{b}e_3(\widehat{\tilde{c}} + \widehat{\tilde{d}}e_3) = \widehat{a}\widehat{g} + \widehat{b}e_3\widehat{\tilde{g}}.$$

One may verify that $\widehat{f * g} = \widehat{f}\widehat{g}$ now holds for example if g is an even function or f is complex-valued.

We begin by finding for each of our partial differential operators ∂_ϕ a corresponding inverse operator T_ϕ which satisfies

$$\partial_\phi T_\phi f = T_\phi \partial_\phi f = f$$

for sufficiently nice functions f. We might as well assume that f lies in C_0^∞ for a moment. The operators T_ϕ happen to be represented by the integrals

$$T_\phi f(h) = \frac{1}{2\pi^2} \int_{\mathbb{H}} f(\omega) \frac{Q_{h-\omega} \cdot \overline{\phi}}{|h-\omega|^4} d\omega.$$

To justify that this gives an actual inverse to ∂_ϕ on C_0^∞ we refer to the following general result found in [Ste70]. To save the reader from a painstaking hunt for a sign error, we remark that Stein uses a definition of the Fourier transform that differs from ours by a minus sign.

Theorem 5.1. *(The Hecke identities). Let $P_k(x) : \mathbb{R}^n \mapsto \mathbb{C}$ be a homogenous harmonic polynomial of degree $k \geq 1$. If $0 \leq \alpha < n$, then the integral operator R given by*

$$Rf(x) = \int_{\mathbb{R}^n} f(y) \frac{P_k(x-y)}{|x-y|^{n+k-\alpha}} dy$$

is on L^2 represented by the multiplier

$$\widehat{Rf}(\xi) = \gamma_{n,k,\alpha} \frac{P_k(\xi)}{|\xi|^{k+\alpha}} \widehat{f}(\xi), \quad \text{where} \quad \gamma_{n,k,\alpha} = (-i)^k \pi^{n/2-\alpha} \frac{\Gamma(k/2+\alpha/2)}{\Gamma(k/2+n/2-\alpha/2)}.$$

In our case we have $n = 4$, and to apply this theorem to T_ϕ we must set $k = \alpha = 1$ so that $\gamma_{n,k,\alpha} = -i\pi$. Observe that any polynomial of degree one is harmonic. Thus

$$\begin{aligned}
(T_\phi f)^\wedge(\xi) &= \left(-\frac{1}{2\pi^2} \int_{\mathbb{H}} f(\omega) \frac{Q_{h-\omega} \cdot \overline{\phi}}{|h-\omega|^4} d\omega \right)^\wedge (\xi) \\
&= \frac{1}{2\pi^2} \sum_{i=1}^{4} \left(\int_{\mathbb{H}} \frac{h_i - \omega_i}{|h-\omega|^4} f(\omega) d\omega \right)^\wedge (\xi) \overline{\phi}_i \\
&= -\frac{1}{2\pi^2} \sum_{i=1}^{4} i\pi \frac{\xi_i}{|\xi|^2} \widehat{f}(\xi) \overline{\phi}_i \\
&= -\frac{i}{2\pi} \widehat{f}(\xi) \frac{Q_\xi \cdot \overline{\phi}}{|\xi|^2} \\
&= \frac{1}{2\pi i} \widehat{f}(\xi) (Q_\xi \cdot \phi)^{-1}.
\end{aligned}$$

Comparing this with (5.3) shows that T_ϕ indeed gives an inverse to ∂_ϕ on C_0^∞. Note also the identity $\mathcal{S}_{\phi\psi} f = \partial_\psi T_\phi f$. The operator T_e usually carries the name Teodorescu transform in the literature, see e.g. [GS98], which is why we will opt to use this name for our operators T_ϕ as well.

We would of course like to pass these kinds of identities to the setting of weak (i.e. distributional) derivatives. Since in general the Leibniz rule fails for the differential operators ∂_ϕ, we will have to think about the definition of weak ∂_ϕ-derivatives for a moment. For $f, g \in C_0^\infty$ we may compute that

$$\int_{\mathbb{H}} (\partial_\phi f) g = \int_{\mathbb{H}} \sum_{k=1}^{4} \frac{\partial f}{\partial h_i} \phi_k g = -\int_{\mathbb{H}} \sum_{k=1}^{4} f \phi_k \frac{\partial g}{\partial h_i} = -\int_{\mathbb{H}} f \partial_\phi^l g,$$

where $\partial_\phi^l g = \sum_{i=1}^4 \phi_i \partial g / \partial h_i$ denotes the 'left'-theory analogue of the differential operator ∂_ϕ, see Remark 5.1. Consequently we are led to define that $f \in L^1_{loc}$ has a weak ∂_ϕ-derivative $\partial_\phi f \in L^1_{loc}$ if for all $g \in C_0^\infty$

$$\int_{\mathbb{H}} (\partial_\phi f) g = -\int_{\mathbb{H}} f \partial_\phi^l g$$

holds. We now collect some related results in the following theorem.

Theorem 5.2. *Let $\phi, \psi \in \mathbb{H}^4$ be orthonormal bases (i.e. satisfying 5.5) as usual. Then we have that*

1. *The Teodorescu transforms T_ϕ have the following mapping properties:*

$$\begin{aligned} T_\phi &: L^1 \mapsto L^{4/3,w} \\ T_\phi &: L^p \mapsto L^{\frac{4p}{4-p}} \text{ for } 1 < p < 4 \\ T_\phi &: L^4 \mapsto VMO \\ T_\phi &: L^p \mapsto C^{1-\frac{4}{p}} \text{ for } 4 < p < \infty, \end{aligned}$$

   where $L^{4/3,w}$ denotes the weak-$L^{4/3}$ space and C^α is the space of Hölder-continuous functions of exponent α.

2. *The operator $\mathcal{S}_{\phi\psi}$ has the (almost) Calderón-Zygmund integral representation as*

$$\mathcal{S}_{\phi\psi} f(h) = \frac{1}{2\pi^2} \int_{\mathbb{H}} f(\omega) \frac{|h-\omega|^2 \overline{\phi} \cdot \psi - 4(Q_{h-\omega} \cdot \overline{\phi})(Q_{h-\omega} \cdot \psi)}{|h-\omega|^6} d\omega + \frac{1}{4} f(h) \overline{\phi} \cdot \psi. \tag{5.6}$$

3. *Let us define the operators T_ϕ^l and $\mathcal{S}_{\phi\psi}^l$ by*

$$T_\phi^l f(h) = \frac{1}{2\pi^2} \int_{\mathbb{H}} \frac{Q_{h-\omega} \cdot \overline{\phi}}{|h-\omega|^4} f(\omega) d\omega$$

and

$$\mathcal{S}^l_{\phi\psi}f(h)=\frac{1}{2\pi^2}\int_{\mathbb{H}}\frac{|h-\omega|^2\psi\cdot\overline{\phi}-4(Q_{h-\omega}\cdot\psi)(Q_{h-\omega}\cdot\overline{\phi})}{|h-\omega|^6}f(\omega)d\omega+\frac{1}{4}(\psi\cdot\overline{\phi})f(h).$$

Then we have the identities

$$\int_{\mathbb{H}}(T_\phi f)g=-\int_{\mathbb{H}}fT^l_\phi g \text{ and } \int_{\mathbb{H}}(\mathcal{S}_{\phi\psi}f)g=\int_{\mathbb{H}}f\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}g$$

valid for $f,g\in C_0^\infty$.

4. *Let* f *be locally integrable with the distributional derivatives* $\partial_\phi f$ *and* $\partial_\psi f$ *in* L^p *for some* $p\in(1,\infty)$. *Then we have the identity*

$$\mathcal{S}_{\phi\psi}\partial_\phi f=\partial_\psi f.$$

Proof. **Proof of 1.** We remark that our proofs of these mapping properties for the Teodorescu transforms T_ϕ follow the same lines of thought as the proofs of the analogous facts for the Cauchy transform in the plane, with our source being the book [AIM09]. The first two of our claimed mapping properties are special cases of the Hardy-Littlewood-Sobolev theorem on Riesz potentials, see [Ste70], page 119. In short, we have that for $1<p<4$

$$\left|\left|\int_{\mathbb{R}^4}|h-\omega|^{-3}f(\omega)d\omega\right|\right|_{\frac{4p}{4-p}}\le E_p||f||_p$$

for some constants E_p. In addition, the mapping property $T_\phi: L^1\mapsto L^{4/3,w}$ follows from the so-called weak part of the same theorem.

We now prove that $T_\phi: L^4\mapsto VMO$. By translation and rescaling it will be enough to estabilish the bound

$$\int_{\mathbb{B}}|T_\phi g(\tau)-(T_\phi g)_{\mathbb{B}}|d\tau\le c||g||_4$$

for some constant c and all $g\in C_0^\infty$. Here $(T_\phi g)_{\mathbb{B}}$ denotes the integral average of $T_\phi g$ over the unit ball $\mathbb{B}=\{h\in\mathbb{H}:|h|<1\}$, which we will have to compute. First we use Fubini's theorem to find that

$$(T_\phi g)_{\mathbb{B}}=\frac{2}{\pi^2}\int_{\mathbb{H}}g(h)\left(\frac{1}{2\pi^2}\int_{\mathbb{B}}\frac{Q_{h-\omega}\cdot\phi}{|h-\omega|^4}d\omega\right)dh=-\frac{2}{\pi^2}\int_{\mathbb{H}}g(h)T_\phi\chi_{\mathbb{B}}(h)dh.$$

This leads us to calculate the Theodorescu transform of the characteristic function of the unit ball. Define the auxiliary function

$$G_\phi(h)=\begin{cases}(Q_h\cdot\overline{\phi})/4, & h\in\mathbb{B}\\ (Q_h\cdot\overline{\phi})/(4|h|^4), & h\in\mathbb{H}\setminus\mathbb{B}\end{cases}.$$

It is then only a matter of calculation to show that $\partial_\phi G_\phi = \chi_{\mathbb{B}}$, yielding the identity

$$(T_\phi g)_{\mathbb{B}} = -\frac{2}{\pi^2}\int_{\mathbb{B}} g(h)\frac{Q_h\cdot\overline{\phi}}{4}dh - \frac{2}{\pi^2}\int_{\mathbb{H}\setminus\mathbb{B}} g(h)\frac{Q_h\cdot\overline{\phi}}{4|h|^4}dh.$$

Hence

$$\begin{aligned}T_\phi g(\tau) - (T_\phi g)_{\mathbb{B}} &= \frac{1}{2\pi^2}\int_{\mathbb{B}} g(h)\left(\frac{Q_{\tau-h}\cdot\overline{\phi}}{|\tau-h|^4} + Q_h\cdot\overline{\phi}\right)dh \\ &\quad + \frac{1}{2\pi^2}\int_{\mathbb{H}\setminus\mathbb{B}} g(h)\left(\frac{Q_{\tau-h}\cdot\overline{\phi}}{|\tau-h|^4} + \frac{Q_h\cdot\overline{\phi}}{|h|^4}\right)dh.\end{aligned}$$

Let us define another auxiliary function Ψ by

$$\Psi(h) = \left\{\begin{array}{l} \frac{1}{2\pi^2}\int_{\mathbb{B}}\left|\frac{Q_{\tau-h}\cdot\overline{\phi}}{|\tau-h|^4} + Q_h\cdot\overline{\phi}\right|d\tau, \quad h\in\mathbb{B} \\ \frac{1}{2\pi^2}\int_{\mathbb{B}}\left|\frac{Q_{\tau-h}\cdot\overline{\phi}}{|\tau-h|^4} + \frac{Q_h\cdot\overline{\phi}}{|h|^4}\right|d\tau, \quad h\in\mathbb{H}\setminus\mathbb{B}\end{array}\right. .$$

After we show that Ψ is in $L^{4/3}$, we may apply Hölder's inequality in the form

$$\int_{\mathbb{B}} |T_\phi g(\tau) - (T_\phi g)_{\mathbb{B}}|d\tau \le \int_{\mathbb{H}} |g(h)||\Psi(h)|dh \le ||g||_4||\Psi||_{4/3},$$

which yields the result. Let us thus prove that $\Psi\in L^{4/3}$. Since the function $|h|^{-3}$ is locally integrable in $\mathbb{H}$, it is not hard to show that $\Psi(h)$ is bounded for small h. Thus it will be enough to prove that $\Psi(h) = \mathcal{O}(|h|^{-4})$ for large h, say $|h|>2$. We compare the integrand in the definition of Ψ to $|h|^{-4}$ and obtain the estimate

$$\begin{aligned}|h|^4\left|\frac{Q_{\tau-h}\cdot\overline{\phi}}{|\tau-h|^4} + \frac{Q_h\cdot\overline{\phi}}{|h|^4}\right| &= \left||h|^4(Q_{\tau-h}\cdot\overline{\phi}) + |\tau-h|^4(Q_h\cdot\overline{\phi})\right||\tau-h|^{-4} \\ &= \left||h|^4(Q_\tau\cdot\overline{\phi}) + (|\tau-h|^4 - |h|^4)(Q_h\cdot\overline{\phi})\right||\tau-h|^{-4} \\ &\le |h|^4|\tau-h|^{-4} + ||\tau-h|^4 - |h|^4||h||\tau-h|^{-4}.\end{aligned}$$

Given $|h|>2$ and $|\tau|<1$, we may find a uniform bound for the last expression since we have $|\tau-h|\ge|h|/2$ and $||\tau-h|-|h||\le 1$. The claim follows.

Let us then move onto the last claim $T_\phi : L^p \mapsto C^{1-\frac{4}{p}}$ for $4<p<\infty$. It is required to show that $|T_\phi g(h) - T_\phi g(\omega)|/|h-\omega|^{1-4/p} \le F_p||g||_p$ for some constant F_p and all functions $g\in L^p$. As the Teodorescu transforms are given in terms of convolution they commute with translations and hence we may assume that $\omega = 0$ without loss of generality. Now the required condition will follow from the estimate

$$\begin{aligned}
\frac{T_\phi g(h) - T_\phi g(0)}{|h|^{1-4/p}} &= \frac{1}{2\pi^2}\frac{1}{|h|^{1-4/p}}\left|\int_{\mathbb{H}} g(\omega)\left(\frac{Q_{h-\omega}\cdot\phi}{|h-\omega|^4} + \frac{Q_\omega\cdot\phi}{|\omega|^4}\right)d\omega\right| \\
&\leq \frac{1}{2\pi^2}||g||_p\left(\int_{\mathbb{H}} |h|^{\frac{4-p}{p-1}}\left|\frac{Q_{h-\omega}\cdot\phi}{|h-\omega|^4} + \frac{Q_\omega\cdot\phi}{|\omega|^4}\right|^{\frac{p}{p-1}} d\omega\right)^{\frac{p-1}{p}} \\
&= \frac{1}{2\pi^2}||g||_p\left(\int_{\mathbb{H}} \frac{|h|^{\frac{4-p}{p-1}}}{|h-\omega|^{\frac{3p}{p-1}}|\omega|^{\frac{3p}{p-1}}}\,|\Phi(h,\omega)|^{\frac{p}{p-1}}\,d\omega\right)^{\frac{p-1}{p}} \\
&= \frac{1}{2\pi^2}||g||_p\left(\int_{\mathbb{H}} \frac{|h|^{\frac{-3p}{p-1}}}{|1-\tau|^{\frac{3p}{p-1}}|\tau|^{\frac{3p}{p-1}}}\,|\Phi(h,h\tau)|^{\frac{p}{p-1}}\,d\tau\right)^{\frac{p-1}{p}} \\
&= \frac{1}{2\pi^2}||g||_p\left(\int_{\mathbb{H}} \frac{1}{|1-\tau|^{\frac{3p}{p-1}}|\tau|^{\frac{3p}{p-1}}}\left|\Phi\left(\frac{h}{|h|},\frac{h}{|h|}\tau\right)\right|^{\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}},
\end{aligned}$$

where

$$\Phi(h,\omega) = (Q_\omega\cdot\phi)(Q_\omega\cdot\overline{\phi})^2 + (Q_{h-\omega}\cdot\phi)(Q_{h-\omega}\cdot\overline{\phi})^2.$$

It remains to show that the integral on the utmost right hand side remains finite and bounded in terms of h. Since $p > 4$ we have that $3p/(p-1) < 4$, so the singularities at $\tau = 0$ and $\tau = 1$ may be overlooked. What is crucial is the behaviour of $\Phi(h/|h|, (h/|h|)\tau)$ when τ is large. It will be enough to prove that $\Phi(h/|h|, (h/|h|)\tau) \leq M|\tau|^2$ for $|\tau| > 1$ and some constant M not depending on h, since then our integrand is of order $|\tau|^{2p/(p-1)}/|\tau|^{6p/(p-1)} = |\tau|^{-4p/(p-1)}$, which is integrable in the region $\{\tau : |\tau| > 1\}$. But

$$\begin{aligned}
\Psi\left(\frac{h}{|h|},\frac{h}{|h|}\tau\right) &= \frac{1}{|h|^3}(Q_{h\tau}\cdot\phi)(Q_{h\tau}\cdot\overline{\phi})^2 \\
&\quad + \frac{1}{|h|^3}(Q_h\cdot\phi - Q_{h\tau}\cdot\phi)(Q_h\cdot\overline{\phi} - Q_{h\tau}\cdot\overline{\phi})^2,
\end{aligned}$$

and if the second term on the right hand side is expanded, we find that the first term cancels out. The rest of the expression only contains terms which are comparable to τ^2, τ or 1, so we are done. Hence we have constants F_p so that the estimate $||T_\phi g||_{C^{1-4/p}} \leq F_p||g||_p$ holds when $4 < p < \infty$.

Proof of 2. The reason for this curious representation is that the polynomial $(Q_h\cdot\overline{\phi})(Q_h\cdot\psi)$ fails to be harmonic in general. A simple calculation, however, shows that

$$|h|^2\overline{\phi}\cdot\psi - 4(Q_h\cdot\overline{\phi})(Q_h\cdot\psi)$$

is harmonic. Thus if F denotes the right hand side of (5.6) we find by

Theorem 5.1 that

$$\begin{aligned}\widehat{F}(\xi) &= \frac{\gamma_{4,2,0}}{2\pi^2}\widehat{f}(\xi)\frac{|\xi|^2\overline{\phi}\cdot\psi - 4(Q_\xi\cdot\overline{\phi})(Q_\xi\cdot\psi)}{|\xi|^2} + \frac{1}{4}\widehat{f}(\xi)\overline{\phi}\cdot\psi \\ &= \widehat{f}(\xi)\frac{(Q_\xi\cdot\overline{\phi})(Q_\xi\cdot\psi)}{|\xi|^2} \\ &= \widehat{\mathcal{S}_{\phi\psi}f}(\xi)\end{aligned}$$

as wanted.

Proof of 3. These two identities are consequences of the fact that T_ϕ and $\mathcal{S}_{\phi\psi}$ have convolution-type repesentations. There should be no trouble formalizing the calculation

$$\begin{aligned}\int T_\phi f(h)g(h)dh &= \frac{1}{2\pi^2}\int_{\mathbb{H}}\int_{\mathbb{H}} f(\omega)\frac{Q_{h-\omega}\cdot\overline{\phi}}{|h-\omega|^4}g(h)d\omega \\ &= -\frac{1}{2\pi^2}\int_{\mathbb{H}}\int_{\mathbb{H}} f(\omega)\frac{Q_{\omega-h}\cdot\overline{\phi}}{|\omega-h|^4}g(h)dhd\omega \\ &= -\int_{\mathbb{H}} f(\omega)T^l_\phi g(\omega)d\omega\end{aligned}$$

and the analogous one for $\mathcal{S}_{\phi\psi}$.

In the sense of Remark 5.1 we actually see that T^l_ϕ is the 'left'-theory analogue of the operator T_ϕ, i.e. it inverts left-derivatives ∂^l_ϕ. In the same sense $\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}$ is the 'left'-analogue of the generalized Beurling transform. These operators naturally inherit the same mapping properties as their 'right'-analogues. Especially the identity $\int(\mathcal{S}_{\phi\psi}f)g = \int f\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}g$ may be generalized to $f\in L^p$ and $g\in L^q$, $1/p+1/q=1$.

Proof of 4. Let $g\in C_0^\infty$ denote a test function. The claim will follow when we verify the identity $\partial^l_\phi\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}g = \partial^l_\psi g$ (by comparing Fourier multipliers for example) and compute that

$$\int(\mathcal{S}_{\phi\psi}\partial_\phi f)g = \int(\partial_\phi f)\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}g = -\int f\partial^l_\phi\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}g = -\int f\partial^l_\psi g = \int(\partial_\psi f)g.$$

□

5.2 Spectral results

Due to various reasons we will need some information about the spectrum of our operators $\mathcal{S}_{\phi\psi}: L^p\mapsto L^p$. Note that $\mathcal{S}_{\phi\psi}$ is complex linear from the left but not the right (i.e. $\mathcal{S}_{\phi\psi}(\zeta f)=\zeta\mathcal{S}_{\phi\psi}f$, but in general $\mathcal{S}_{\phi\psi}(f\zeta)\neq(\mathcal{S}_{\phi\psi}f)\zeta$). This leads us to consider $\mathcal{S}_{\phi\psi}$ as an element of the Banach algebra $\mathcal{B}_p$ of operators on $L^p(\mathbb{H},\mathbb{H})$ that are complex-linear from the left.

Theorem 5.3. *Let T be an operator in $\mathcal{B}_2$ represented by the right-multiplier*

$$\widehat{Tf}(\xi) = \widehat{f}(\xi)m(\xi),$$

where m is a quaternion-valued function that is homogeneous of degree zero and is in the class C^3. Assume further that $|m| = 1$ almost everywhere. Then T extends to a bounded operator on L^p for $1 < p < \infty$ and the spectrum of T with respect to $\mathcal{B}_p$ defined by

$$\sigma(T) = \{\zeta \in \mathbb{C} : f \mapsto Tf - \zeta f \text{ is not invertible on } L^p(\mathbb{H}, \mathbb{H})\}$$

satisfies

$$\sigma(T) \subset \{\zeta \in \mathbb{C} : |\zeta| = 1\}.$$

Proof. The proof will rely on some knowledge of when multiplier operators given on L^2 extend to bounded operators on L^p. Namely, we will use the Mikhlin multiplier theorem (see [Ste70], page 96) which states that if there exists a constant $B > 0$ such that

$$\left|\frac{\partial^\alpha m}{\partial h_1^{\alpha_1} \cdots \partial h_4^{\alpha_4}}(\xi)\right| \leq B|\xi|^{-|\alpha|} \text{ when } |\alpha| \leq 3, \tag{5.7}$$

then the operator T given by the multiplier m on $L^2(\mathbb{R}^4)$ is extendable to a bounded operator on $L^p(\mathbb{R}^4)$ for $1 < p < \infty$. Of course, this result is originally not proven for quaternion-valued functions but by passing to components one may generalize the result to our setting.

The assumptions that m is homogeneous of degree zero and in C^3 guarantee that the condition (5.7) is satisfied, and hence the operator T as given in the theorem extends to be bounded on L^p. The same can be said for the operator T^{-1} which, by the condition $|m| = 1$, is given by the right-multiplier $\overline{m}$.

For our claim about the spectrum of T we have to prove that the operator $T - \zeta$ is invertible whenever $|\zeta| \neq 1$. Let us begin by noting the identity

$$(T - \zeta)(T^{-1} - \zeta) = (T^{-1} - \zeta)(T - \zeta) = 1 + \zeta^2 - \zeta(T + T^{-1}),$$

which shows that it will be enough to estabilish the invertibility of the operator $1 + \zeta^2 - \zeta(T + T^{-1})$. Working first on L^2, we calculate the multiplier of this operator by

$$\begin{aligned}((1 + \zeta^2)f - \zeta(Tf + T^{-1}f))^\wedge(\xi) &= (1 + \zeta^2)\widehat{f}(\xi) - \zeta\widehat{f}(\xi)(m(\xi) + \overline{m(\xi)}) \\ &= ((1 + \zeta^2) - 2\zeta \operatorname{Re} m(\xi))\widehat{f}(\xi),\end{aligned}$$

Due to this representation it will be enough to prove that the operator on L^2 given by the left-multiplier

$$m_\zeta(\xi) = ((1 + \zeta^2) - 2\zeta \operatorname{Re} m(\xi))^{-1}$$

extends to a bounded operator on L^p for $1 < p < \infty$. Observe that the multiplier m_ζ is again homogeneous of degree zero, and will be in C^3 as soon as we show that $(1+\zeta^2) - 2\zeta \operatorname{Re} m(\xi)$ is nonzero. Supposing m_ζ vanishes leads to the equation

$$\zeta^{-1} + \zeta = 2 \operatorname{Re} m,$$

which would imply that the quantity $\zeta^{-1} + \zeta$ is a real number. It is easy to check that this expression is real exactly when $|\zeta| = 1$ or when $\zeta \in \mathbb{R}$. In the latter case we find the estimate

$$|\zeta^{-1} + \zeta| > 2 \geq 2|\operatorname{Re} m|,$$

yielding a contradiction and hence our claim. □

Remark 5.2. The true value of this theorem is that it implies the spectral radius of the operator T to equal one. By the spectral radius theorem this further implies that $\lim_{n\to\infty} ||T^n||_p^{1/n} = 1$. Now given ϕ and ψ that satisfy the orthonormality condition (5.5) we see that the right-multiplier $(Q_\xi \cdot \overline{\phi})(Q_\xi \cdot \psi)/|\xi|^2$ of the generalized Beurling transform $\mathcal{S}_{\phi\psi}$ satisfies the conditions imposed on the multiplier m in the previous theorem. Hence the statement of the theorem remains valid for $\mathcal{S}_{\phi\psi}$ in place of T.

Note also that it was the right choice to consider the spectrum of $\mathcal{S}_{\phi\psi}$ with respect to left-multiplication by complex numbers, since $\mathcal{S}_{\phi\psi}$ was only complex linear from the left. One might be skeptic as to whether there is any value in calculating the spectrum with respect to right-multiplication, but we have done it nonetheless. In fact, we conjecture the following result to be a consequence of a general solvability result for quaternionic Beltrami-type equations, but have only been able to prove this special case.

Theorem 5.4. *The right-spectrum of* $\mathcal{S}_{\phi\psi}$

$$\sigma^r(\mathcal{S}_{\phi\psi}) = \{\zeta \in \mathbb{C} : f \mapsto \mathcal{S}_{\phi\psi} f - f\zeta \text{ is not invertible on } L^p(\mathbb{H},\mathbb{H})\}$$

satisfies

$$\sigma^r(\mathcal{S}_{\phi\psi}) \subset \{\zeta \in \mathbb{C} : |\zeta| = 1\}.$$

Proof. Let us first solve the equation

$$\partial_\psi f - \partial_\phi f \zeta = H,$$

where we assume that ζ is a complex number with $|\zeta| \neq 1$. For simplicity let us assume $H \in C_0^\infty$. The idea is to make the substitution $f(h) = g(Lh)$, where $L = (L_{ij})$ is an invertible 4-by-4 real matrix that is chosen so that this reduces to the equation $\partial_e g \circ L = H$. Let us first compute

$$\partial_\psi f(h) = \sum_{i=1}^{4} \frac{\partial}{\partial h_i} g(Lh)\psi_i = \sum_{i,j=1}^{4} \frac{\partial g}{\partial h_j}(Lh) L_{ji}\psi_i.$$

From this it follows that

$$\partial_\psi f - \partial_\phi f \zeta = \sum_{j=1}^4 \frac{\partial g}{\partial h_j}(Lh)\left(\sum_{i=1}^4 L_{ji}\psi_i - L_{ji}\phi_i\zeta\right).$$

We would like the last expression to equal $\partial_e g(Lh)$, or equivalently

$$\sum_{i=1}^4 L_{ji}\psi_i - L_{ji}\phi_i\zeta = e_j$$

for $j = 1, \ldots, 4$. Write $L^j = (L_{j1}, L_{j2}, L_{j3}, L_{j4})$ so that the above becomes

$$L^j \cdot \psi - (L^j \cdot \phi)\zeta = e_j.$$

That each of these equations has a solution L^j now follows from the fact that the linear transformation $R : \mathbb{R}^4 \mapsto \mathbb{H}$ given by

$$Rx = x \cdot \psi - (x \cdot \phi)\zeta$$

is injective since $|\zeta| \neq 1$ and thus surjective as well. Note also that by the construction $R \circ L$ is the identity map and thus L is invertible as well. We have shown that the equation $\partial_\psi f - \partial_\phi f \zeta = H$ is equivalent to $(\partial_e g) \circ L = H$ and hence to $\partial_e g = H \circ R$. This we can solve when H is nice enough, as a solution is given by $g = T_e(H \circ R)$.

If one now wants to show that the right-spectrum of $\mathcal{S}_{\phi\psi}$ on L^p is contained in the unit circle, it is enough to show that the operator $\Xi = \Xi(\phi, \psi, \zeta)$ given by

$$\Xi\omega = \mathcal{S}_{\phi\psi}\omega - \omega\zeta$$

is invertible on L^p for all $\zeta \in \mathbb{C}$ with $|\zeta| \neq 1$. We will actually compute the inverse of this operator. It will be useful to think of this as solving

$$\mathcal{S}_{\phi\psi}\omega - \omega\zeta = H \tag{5.8}$$

uniquely for ω and for each $H \in L^p$. Assume for a moment that $H \in C_0^\infty$. Let f denote the corresponding solution to $\partial_\psi f - \partial_\phi f \zeta = H$. Then $\omega = \partial_\phi f$ solves (5.8). We now write $\partial_\phi f$ as a bounded operator of H, which is justified by the fact that f is constructed in terms of H. Recall that

$$\partial_\phi f = \sum_{j=1}^4 \left(\frac{\partial g}{\partial h_j} \circ L\right)(L^j \cdot \phi),$$

where g solves $\partial_e g = H \circ L^{-1}$. We now express each partial derivative $\partial g/\partial h_j$ in terms of bounded operators of $\partial_e g$. This may be done for example by defining the quadruplets

$$\rho_1 = (1,0,0,0)$$
$$\rho_2 = (0,1,0,0)$$
$$\rho_3 = (0,0,1,0)$$
$$\rho_4 = (0,0,0,1)$$

so that $\partial g/\partial h_j = \partial_{\rho_j} g = \mathcal{S}_{e\rho_j}\partial_e g$ for $j = 1,...,4$. Note that these quadruplets do not satisfy the orthogonality condition 5.5, but that this was not necessary to define the Beurling transform. Especially we find that

$$\begin{aligned}\partial_\phi f &= \sum_{j=1}^{4}\left(\frac{\partial g}{\partial h_j}\circ L\right)(L^j\cdot\phi)\\ &= \sum_{j=1}^{4}((\partial_{\rho_j} g)\circ L)(L^j\cdot\phi)\\ &= \sum_{j=1}^{4}(\mathcal{S}_{e\rho_j}(H\circ L^{-1})\circ L)(L^j\cdot\phi).\end{aligned}$$

So if T_ζ denotes the operator given by the change of variables $T_\zeta g(h) = g(Lh)$ then

$$H = \Xi\left[\sum_{j=1}^{4}(T_\zeta \mathcal{S}_{e\rho_j} T_\zeta^{-1} H)(L^j\cdot\phi)\right] \tag{5.9}$$

at least for $H \in C_0^\infty$. Now we may generalize this identity to $H \in L^p$ by density and the boundedness of the operator on the right hand side. We have proven the right-invertibility of Ξ. The left-invertibility may be proven by arguing that the operators

$$\Xi(H) \quad \text{and} \quad \sum_{j=1}^{4}(T_\zeta \mathcal{S}_{e\rho_j} T_\zeta^{-1} H)(L^j\cdot\phi)$$

are represented by right-Fourier multipliers and by (5.9) these multipliers are mutual inverses. Thus these two operators commute. □

5.3 Quaternionic Beltrami equations

Let us now state and solve our generalization of the complex Beltrami equation: The equation we solve is

$$\partial_\phi f(h) = \mu(h)\partial_\psi f(h) + H(h),$$

where $H \in L^p(\mathbb{H},\mathbb{H})$. This may again be reduced to the equation

$$\omega(h) = \mu(h)(\mathcal{S}_{\phi\psi}\omega)(h) + H(h). \tag{5.10}$$

Here μ is a quaternion-valued function, and we identify μ with the operator defined by left-multiplication by μ. This reduces (5.10) to the invertibility of $1 - \mu\mathcal{S}_{\phi\psi}$. The strongest assumption we will invert this operator under is that μ is a compactly supported function in VMO with $||\mu||_\infty = k < 1$.

Of course, we again need a compactness result for the commutators of μ with $\mathcal{S}_{\phi\psi}$. Recalling the original definition of $\mathcal{S}_{\phi\psi}$,

$$\mathcal{S}_{\phi\psi} = -(\mathcal{R} \cdot \psi)(\mathcal{R} \cdot \overline{\phi}),$$

it would be enough to verify that commutators of VMO-functions with the four-dimensional Riesz transforms are compact. This is indeed true, and it is proven in the exact same way as Theorem 4.2. We skip the proof for brevity, and instead refer the more curious reader to the original article by Uchiyama, [Uch78], where the same result is proven for general Calderón-Zygmund integral operators.

Recall once more that the proof of the invertibility of the basic Beltrami operator consisted of

- For $p \geq 2$:
  1. Showing that our operator is Fredholm by a Neumann-series type argument.
  2. Continuously deforming our operator to the identity to show that the Fredholm index is zero.
  3. Applying a L^2-estimate to show that the kernel is trivial.
- For $p \leq 2$.
  1. Verifying that the transpose on L^q is invertible, and finishing by duality.

Now for $p \geq 2$ the first step is essentially done, since by the compactness result mentioned earlier the operators $(\mu\mathcal{S}_{\phi\psi})^n$ and $\mu^n\mathcal{S}_{\phi\psi}^n$ are equivalent modulo compact operators, and due to Remark 5.2 we find that for any $\epsilon > 0$ and large n the estimate $||\mathcal{S}_{\phi\psi}^n||_p \leq (1+\epsilon)^n$ holds. Choosing ϵ so that $k(1+\epsilon) < 1$ shows that $\lim_{n\to\infty} ||\mu^n\mathcal{S}_{\phi\psi}^n||_p = 0$ as wanted. For step two one replaces μ with $t\mu$ for $t \in [0,1]$, and for step three we argue as usual. Thus the case $p \geq 2$ is handled.

Remark 5.3. One might ask whether an analogue of Theorem 4.4 would hold in the quaternionic setting, giving a polynomial bound on the growth of the norms $||\mathcal{S}_{\phi\psi}^n||_p$ in terms of n and thus sparing us from finding the spectrum of $\mathcal{S}_{\phi\psi}$. We, however, find it unlikely that such an analogue would be easy to prove as the proof in our source, [AIM09], was by itself quite lengthy and relied heavily on the fact that the Calderón-Zygmund integral representations of the associated iterated operators were easy to compute.

And as we have seen in part 2. of Theorem 5.2, the Calderón-Zygmund representations of the analogous quaternionic operators may be considerably harder to compute due to certain polynomials failing to be harmonic and thus making it harder to use Theorem 5.1.

Perhaps the most interesting step in proving the invertibility of $1 - \mu\mathcal{S}_{\phi\psi}$ is the duality argument, for which we hence let $1 < p \leq 2$. We consider for simplicity the dual space of continuous linear functionals $F : L^p(\mathbb{H}, \mathbb{H}) \mapsto \mathbb{R}$. Note that by Theorem 4.8 it is clear that the dual is $L^q(\mathbb{H}, \mathbb{H})$, with one duality pairing given by

$$F(f) = \langle f, g \rangle = \operatorname{Re} \int_{\mathbb{H}} f(h)g(h)dh,$$

where g is some function in L^q. We now calculate the transpose of our operator $1 - \mu\mathcal{S}_{\phi\psi}$. In view of 3. in Theorem 5.2 and the identity $\operatorname{Re} h_1 h_2 = \operatorname{Re} h_2 h_1$ we see that

$$\begin{aligned}
\operatorname{Re} \int_{\mathbb{H}} (1 - \mu\mathcal{S}_{\phi\psi}) f(h) g(h) dh &= \operatorname{Re} \int_{\mathbb{H}} fg - \operatorname{Re} \int_{\mathbb{H}} \mu\mathcal{S}_{\phi\psi} fg \\
&= \operatorname{Re} \int_{\mathbb{H}} fg - \operatorname{Re} \int_{\mathbb{H}} \mathcal{S}_{\phi\psi} fg\mu \\
&= \operatorname{Re} \int_{\mathbb{H}} f(g - \mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}(g(h)\mu)), \\
&= \operatorname{Re} \int_{\mathbb{H}} f(1 - \mathcal{S}^l_{\overline{\psi}\,\overline{\phi}} r(\mu)) g,
\end{aligned}$$

where we have denoted by $r(\mu)$ the operator given by right-multiplication by μ. Hence the transpose of $1 - \mu\mathcal{S}_{\phi\psi}$ is

$$1 - \mathcal{S}^l_{\overline{\psi}\,\overline{\phi}} r(\mu) = \mathcal{S}^l_{\overline{\psi}\,\overline{\phi}} (1 - r(\mu)\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}})(\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}})^{-1},$$

and for the solvability of (5.10) it remains to argue that $1 - r(\mu)\mathcal{S}^l_{\overline{\psi}\,\overline{\phi}}$ is invertible on L^q, $q \geq 2$, due to the analogy between the 'left'- and 'right'-theories.

For now, this concludes our study of the Beltrami equation in the quaternionic setting, and with it the whole thesis. But we believe that there is much improvement and generalization to be found in the quaternionic (and Clifford-algebraic) settings, at least as far as Beltrami equations are concerned. Indeed, the study of Beltrami-type equations in this context is still in its early stages, and the unification between the geometric point of view and the equations still remains to be found. As a closing remark we would like to note that the restriction to quaternions here was only for simplicity, and that the results and proofs we have stated here should be equally valid in the setting of general Clifford algebras.

6 Appendix

6.1 Notation and definitions

L^p-spaces. Let (Ω, μ) be a measure space and $\mathbb{E}$ be a finite dimensional inner product space whose norm we denote by $|\cdot|$. For each $p \in [1, \infty)$ we define the *L^p-space* $L^p(\Omega, \mathbb{E})$ as the space of measurable functions $f : \Omega \mapsto \mathbb{E}$ for which the L^p-norm

$$||f||_p = \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}}$$

is finite. In addition, the space $L^\infty(\Omega, \mathbb{E})$ consists of all essentially bounded $f : \Omega \mapsto \mathbb{E}$, imbued with the norm $||f||_\infty = \operatorname{ess\,sup}_{x \in \Omega} |f(x)|$. As soon as we quotient out the functions that vanish μ-almost everywhere the spaces $L^p(\Omega, \mathbb{E})$ become Banach spaces for all $p \in [1, \infty]$. For us Ω and $\mathbb{E}$ will simply be instances of $\mathbb{R}^n$ or $\mathbb{C}^n$, and when the context is clear we might write L^p instead of $L^p(\Omega, \mathbb{E})$.

Hölder conjugates. If $p \in [1, \infty]$ we always denote by q its *Hölder conjugate*, i.e. the unique number $q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Characteristic functions. Given a set E, the *characteristic function of E*, denoted by χ_E, is defined to take the value 1 in E and 0 outside of E.

Test functions. The *space of test functions*, C_0^∞, consists of all compactly supported functions f that are smooth, i.e. admit partial derivatives up to all orders. The domain and range of these functions will vary depending on the context.

Distributional derivatives. Given a locally integrable function f, we say that another locally integrable function f_{x_j} is the *distributional derivative* of f with respect to x_j if

$$\int f_{x_j} g = - \int f g_{x_j}$$

for all test functions g (with the domain and range of f). The integral is taken over the domain of f.

References

[Ahl66] L. Ahlfors. *Lectures on Quasiconformal Mappings.* Van Nostrand, Princeton, 1966.

[AIM09] K. Astala, T. Iwaniec, and G. Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane.* Princeton University Press, 2009.

[BJ08] R. Bañuelos and P. Janakiraman. L^p-bounds for the Beurling-Ahlfors transform. *Transactions of the American Mathematical Society*, 360, 2008.

[Con90] J. B. Conway. *A Course in Functional Analysis.* Springer, 1990.

[CRW76] R. R. Coifman, R. Rochberg, and G. Weiss. Factorization theorems for Hardy spaces in several variables. *Annals of Mathematics*, 103, 1976.

[GS98] K. Gürlebeck and W. Sprössig. *Quaternionic and Clifford calculus for physicists and engineers.* Wiley, 1998.

[IM96] T. Iwaniec and G. Martin. Riesz transforms and related singular integrals. *Journal für die reine und angewandte Mathematik*, 473, 1996.

[Käh00] U. Kähler. On quaternionic Beltrami equations. *In: Clifford Algebras and their Applications in Mathematical Physics, Volume II, edited by J. Ryan and W. Sprössig, Birkhäuser, Basel*, 2000.

[Kos10] A. Koski. *The Hilbert transform.* University of Helsinki, Candidate thesis, 2010.

[MV06] J. Mateu and J. Verdera. L^p and weak L^1 estimates for the maximal Riesz transform and the maximal Beurling transform. *Mathematical Research Letters*, 13, 2006.

[Pic72] S. Pichorides. On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov. *Studia Math*, 44, 1972.

[Rud66] W. Rudin. *Real and complex analysis.* McGraw-Hill, 1966.

[Ste70] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, 1970.

[Uch78] A. Uchiyama. On the compactness of operators of Hankel type. *Tohoku Mathematical Journal*, 30, 1978.