Analysis and Geometry of Elastic Deformations

Aleksis Koski

Aalto-yliopisto

Suomen Akatemia

GME Geometric Methods in Elasticity

A brief overview of history

Some key historical highlights in the development of elasticity:

1678, The linear law of elasticity aka **Hooke's law:** Force is proportional to displacement.

A brief overview of history

Some key historical highlights in the development of elasticity:

1678, The linear law of elasticity aka **Hooke's law:** Force is proportional to displacement.

1700's, Euler & Bernoulli: Bending of beams, buckling formula, differential equations.

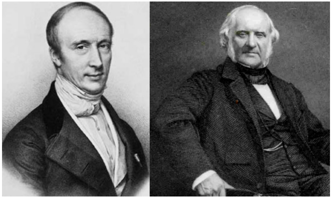
A brief overview of history

Late 1700's, Lagrange: Provided variational formalism, preparing ground for later models.

A brief overview of history

Late 1700's, Lagrange: Provided variational formalism, preparing ground for later models.

1800's, Cauchy & Green: Stress tensor, motion/equilibrium equations. Potential theory, boundary value problems.



A brief overview of history

The modern framework of Nonlinear Elasticity (calc. var. & nonlinear PDE's) was built in the late 1900's, particularly by

A brief overview of history

The modern framework of Nonlinear Elasticity (calc. var. & nonlinear PDE's) was built in the late 1900's, particularly by

Antman

- Nonlinear theory of strings, rods, shells, and 3D bodies.
- Geometric and mechanical realism.

A brief overview of history

The modern framework of Nonlinear Elasticity (calc. var. & nonlinear PDE's) was built in the late 1900's, particularly by

Antman

- Nonlinear theory of strings, rods, shells, and 3D bodies.
- Geometric and mechanical realism.

Ball

- Introduced **polyconvexity** to guarantee existence of minimizers.
- Analysis of singularities, cavities, and microstructure.

A brief overview of history

The modern framework of Nonlinear Elasticity (calc. var. & nonlinear PDE's) was built in the late 1900's, particularly by

Antman

- Nonlinear theory of strings, rods, shells, and 3D bodies.
- Geometric and mechanical realism.

Ball

- Introduced **polyconvexity** to guarantee existence of minimizers.
- Analysis of singularities, cavities, and microstructure.

Ciarlet

- Dimension reduction, Γ-convergence.
- Foundational work in finite element methods.

Nonlinear Elasticity

Given a reference body $\mathbb{X} \subset \mathbb{R}^n$, the **elastic/strain energy** of a deformation $h : \mathbb{X} \to \mathbb{R}^n$ is defined as

$$\mathbb{E}[h] = \int_{\mathbb{X}} W(Dh(x))\, dx,$$

where $W(\cdot)$ is the **stored energy function**, expressing the properties of the material being deformed.

Nonlinear Elasticity

Given a reference body $\mathbb{X} \subset \mathbb{R}^n$, the **elastic/strain energy** of a deformation $h : \mathbb{X} \to \mathbb{R}^n$ is defined as

$$\mathbb{E}[h] = \int_{\mathbb{X}} W(Dh(x))\, dx,$$

where $W(\cdot)$ is the **stored energy function**, expressing the properties of the material being deformed.

Variational formulation: Given an appropriate set of elastic deformations h, find a minimizer for the energy $\mathbb{E}[h]$.

Elastic deformations

Q: What natural constraints should be placed on deformations?

Elastic deformations

Q: What natural constraints should be placed on deformations?

Non-interpenetration of matter: Conditions to ensure that deformations are physically meaningful, e.g.

Elastic deformations

Q: What natural constraints should be placed on deformations?

Non-interpenetration of matter: Conditions to ensure that deformations are physically meaningful, e.g.

- Deformations are injective.

Elastic deformations

Q: What natural constraints should be placed on deformations?

Non-interpenetration of matter: Conditions to ensure that deformations are physically meaningful, e.g.

- Deformations are injective.
- Deformations satisfy $\det Dh > 0$ or ≥ 0.

Elastic deformations

Q: What natural constraints should be placed on deformations?

Non-interpenetration of matter: Conditions to ensure that deformations are physically meaningful, e.g.

- Deformations are injective.
- Deformations satisfy det $Dh > 0$ or ≥ 0.
- Deformations are monotone maps.

Elastic deformations

Q: What natural constraints should be placed on deformations?

Non-interpenetration of matter: Conditions to ensure that deformations are physically meaningful, e.g.

- Deformations are injective.
- Deformations satisfy $\det Dh > 0$ or ≥ 0.
- Deformations are monotone maps.

Analytical requirements: Deformations must have finite energy, $\mathbb{E}[h] < \infty$, requiring e.g.

- Deformations must lie in a Sobolev space $W^{1,p}(\mathbb{X})$.

Elastic deformations

Q: What natural constraints should be placed on deformations?

Non-interpenetration of matter: Conditions to ensure that deformations are physically meaningful, e.g.

- Deformations are injective.
- Deformations satisfy $\det Dh > 0$ or ≥ 0.
- Deformations are monotone maps.

Analytical requirements: Deformations must have finite energy, $\mathbb{E}[h] < \infty$, requiring e.g.

- Deformations must lie in a Sobolev space $W^{1,p}(\mathbb{X})$.

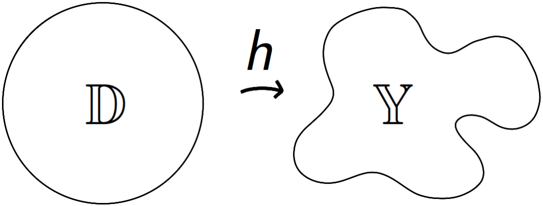
Along with **boundary constraints** (such as fixed boundary values $h|_{\partial\mathbb{X}} = \varphi$), these give rise to various meaningful classes of *elastic deformations* to study.

Example model

Minimize

$$\mathbb{E}_2[h] = \int_{\mathbb{D}} |Dh(z)|^2 \, dz$$

among $W^{1,2}$-homeomorphisms $h : \mathbb{D} \to \mathbb{Y}$, where $\mathbb{D} \subset \mathbb{R}^2$ is the unit disk and $\mathbb{Y}$ is a planar simply connected domain.

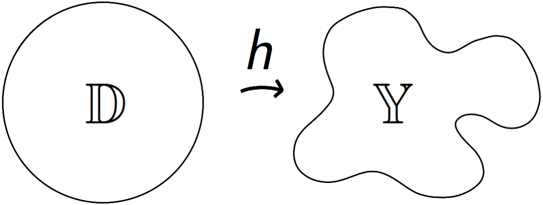


Example model

Minimize

$$\mathbb{E}_2[h] = \int_{\mathbb{D}} |Dh(z)|^2 \, dz$$

among $W^{1,2}$-homeomorphisms $h : \mathbb{D} \to \mathbb{Y}$, where $\mathbb{D} \subset \mathbb{R}^2$ is the unit disk and $\mathbb{Y}$ is a planar simply connected domain.



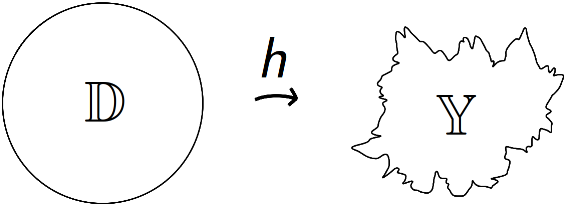
Solution: Minimizer is the conformal map $g : \mathbb{D} \to \mathbb{Y}$ given by the Riemann Mapping Theorem.

I: Do deformations exist?

Elastic deformations

In general, it can be a rather deep question whether admissible deformations even exist.

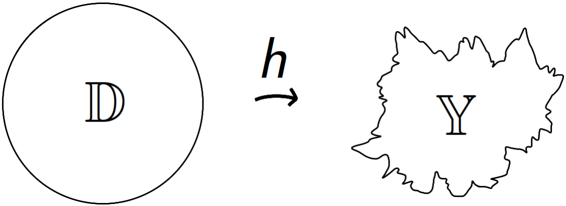
Question: Let $1 \leq p < \infty$. Given a Jordan domain $\mathbb{Y} \subset \mathbb{R}^2$, does there exist a $W^{1,p}$-homeomorphism $h : \mathbb{D} \to \mathbb{Y}$?



Elastic deformations

In general, it can be a rather deep question whether admissible deformations even exist.

Question: Let $1 \leq p < \infty$. Given a Jordan domain $\mathbb{Y} \subset \mathbb{R}^2$, does there exist a $W^{1,p}$-homeomorphism $h : \mathbb{D} \to \mathbb{Y}$?



Answer:

- Riemann Mapping Theorem: Guarantees existence for $p \leq 2$.
- Characterizing existence for $p > 2$ is an open problem.

Elastic deformations

For fixed boundary values, this problem ties in to *Sobolev trace theory*. Let $\mathbb{Y}$ again be a planar Jordan domain, and consider:

Question: Let $1 \leq p \leq \infty$, and let $\varphi : \partial\mathbb{D} \to \partial\mathbb{Y}$ be a given boundary homeomorphism. Suppose that φ is the trace of some $W^{1,p}$-map.

Does φ admit a homeomorphic $W^{1,p}$-extension?

Elastic deformations

For fixed boundary values, this problem ties in to *Sobolev trace theory*. Let $\mathbb{Y}$ again be a planar Jordan domain, and consider:

Question: Let $1 \leq p \leq \infty$, and let $\varphi : \partial\mathbb{D} \to \partial\mathbb{Y}$ be a given boundary homeomorphism. Suppose that φ is the trace of some $W^{1,p}$-map.

Does φ admit a homeomorphic $W^{1,p}$-extension?

The answer depends intricately on the geometry of the target domain $\mathbb{Y}$.

The general answer remains *no* (Zhang 2019, K-Onninen 2023), but conditions under which the answer is positive has been an active topic of research.

Trace theorem for Sobolev homeomorphisms

Let $1 < p < \infty$. Gagliardo's trace theorem says that a map φ defined on $\partial\mathbb{D}$ is in the trace space of $W^{1,p}(\mathbb{D})$ if

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^p}\,dx\,dy \quad < \quad \infty.$$

Trace theorem for Sobolev homeomorphisms

Let $1 < p < \infty$. Gagliardo's trace theorem says that a map φ defined on $\partial\mathbb{D}$ is in the trace space of $W^{1,p}(\mathbb{D})$ if

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^p}\,dx\,dy \quad < \quad \infty.$$

Surprisingly, there is a direct analogue for $W^{1,p}$-homeomorphisms.

Theorem (K-Onninen-Xu 2025)

Let $1 < p < \infty$ and $\varphi : \partial\mathbb{D} \to \partial\mathbb{Y}$ be a boundary homeomorphism. Then φ admits a homeomorphic $W^{1,p}$-extension to $\mathbb{D}$ if and only if

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \frac{d_{\mathbb{Y}}(\varphi(x),\varphi(y))^p}{|x-y|^p}\,dx\,dy \quad < \quad \infty.$$

Here $d_{\mathbb{Y}}$ denotes the internal distance in the domain $\mathbb{Y}$, i.e. length of the shortest connecting curve between two points.

Approximation by diffeomorphisms

Ball-Evans: Is it possible to approximate $W^{1,p}$-homeomorphisms via diffeomorphisms? Main open case: 3D.

Approximation by diffeomorphisms

Ball-Evans: Is it possible to approximate $W^{1,p}$-homeomorphisms via diffeomorphisms? Main open case: 3D.

2D proofs (Iwaniec-Kovalev-Onninen 2011, Hencl-Pratelli 2015) are based on extension methods for Sobolev homeomorphisms.

Approximation by diffeomorphisms

Ball-Evans: Is it possible to approximate $W^{1,p}$-homeomorphisms via diffeomorphisms? Main open case: 3D.

2D proofs (Iwaniec-Kovalev-Onninen 2011, Hencl-Pratelli 2015) are based on extension methods for Sobolev homeomorphisms.

Even the simplest 3D problem of characterizing when a homeomorphism $\varphi : S^2 \to S^2$ can be extended as a $W^{1,p}$-homeomorphism to B^3 remains open.

(Sufficient conditions, Hencl-K-Onninen 2024)

II: Does a minimizer exist?

Well-posedness of the minimization problem

The standard scheme to ensure the existence of a minimizing deformation requires the **weak lower semicontinuity** of the elastic energy $\mathbb{E}[h]$.

Well-posedness of the minimization problem

The standard scheme to ensure the existence of a minimizing deformation requires the **weak lower semicontinuity** of the elastic energy $\mathbb{E}[h]$.

The essential convexity condition to ensure this is Morrey's **quasiconvexity**:

$$W(A) \leq \frac{1}{|\mathbb{X}|} \int_{\mathbb{X}} W(A + Dg)\, dx, \qquad \text{where } g \in C_0^\infty(\mathbb{X}).$$

Quasiconvexity sits neatly between **rank-one convexity** (too weak) and **polyconvexity** (often too strong).

Well-posedness of the minimization problem

The standard scheme to ensure the existence of a minimizing deformation requires the **weak lower semicontinuity** of the elastic energy $\mathbb{E}[h]$.

The essential convexity condition to ensure this is Morrey's **quasiconvexity**:

$$W(A) \leq \frac{1}{|\mathbb{X}|} \int_{\mathbb{X}} W(A + Dg)\, dx, \qquad \text{where } g \in C_0^\infty(\mathbb{X}).$$

Quasiconvexity sits neatly between **rank-one convexity** (too weak) and **polyconvexity** (often too strong).

Morrey's conjecture: Does rank-one convexity imply quasiconvexity (in 2D)?

The Burkholder functional

The most important rank-one convex functional in 2D is the **Burkholder functional**

$$\mathbf{B}_p(A) \equiv \left[\left(\frac{p}{2} - 1\right)|A|^2 - \frac{p}{2}\det A\right]|A|^{p-2}, \qquad A \in \mathbb{R}^{2\times 2}.$$

Proving the quasiconvexity of $\mathbf{B}_p$ would, for example, solve the long-standing open problem of determining the L^p-norms of the Beurling-Ahlfors transform.

The Burkholder functional

The most important rank-one convex functional in 2D is the **Burkholder functional**

$$\mathbf{B}_p(A) \equiv \left[\left(\frac{p}{2} - 1\right)|A|^2 - \frac{p}{2}\det A\right]|A|^{p-2}, \qquad A \in \mathbb{R}^{2\times 2}.$$

Proving the quasiconvexity of $\mathbf{B}_p$ would, for example, solve the long-standing open problem of determining the L^p-norms of the Beurling-Ahlfors transform.

Note that $\mathbf{B}_2(A) = -\det A$.

The Burkholder functional

Generalizing the work of Astala-Iwaniec-Saksman-Prause:

Theorem (Astala-Faraco-Guerra-K-Kristensen 2024)

Let $p \geq 2$, $A \in \mathbb{R}^{2\times 2}$, and $f \in A + W_0^{1,2}(\mathbb{X})$ be such that $\mathbf{B}_p(Df) \leq 0$ a.e. in $\mathbb{X}$. Then

$$\mathbf{B}_p(A) \leq \frac{1}{|\mathbb{X}|} \int_{\mathbb{X}} \mathbf{B}_p\left(Df(z)\right) dz.$$

Note that $\mathbf{B}_p(A) \leq 0$ iff A is $\frac{p}{p-2}$-quasiconformal.

The Burkholder functional

Generalizing the work of Astala-Iwaniec-Saksman-Prause:

Theorem (Astala-Faraco-Guerra-K-Kristensen 2024)

Let $p \geq 2$, $A \in \mathbb{R}^{2\times 2}$, and $f \in A + W_0^{1,2}(\mathbb{X})$ be such that $\mathbf{B}_p(Df) \leq 0$ a.e. in $\mathbb{X}$. Then

$$\mathbf{B}_p(A) \leq \frac{1}{|\mathbb{X}|} \int_{\mathbb{X}} \mathbf{B}_p\left(Df(z)\right) dz.$$

Note that $\mathbf{B}_p(A) \leq 0$ iff A is $\frac{p}{p-2}$-quasiconformal.

Furthermore, the problem

$$\inf \left\{ \int_{\mathbb{X}} \mathbf{B}_p\left(Df(z)\right) dz : f \in g + W_0^{1,p}(\mathbb{X}) \text{ is } K\text{-quasiregular} \right\}$$

admits a minimizer when $2 \leq p < \frac{2K}{K-1}$ and g is K-quasiregular.

III: What does the minimizer look like?

Basic questions to understand about minimizers

Basic questions to understand about minimizers

Regularity:

- How smooth are minimizers?
- Can there be loss of continuity (when $p < 2$)?

Basic questions to understand about minimizers

Regularity:

- How smooth are minimizers?
- Can there be loss of continuity (when $p < 2$)?

Topology (loss of injectivity):

- 2D, $p \geq 2$: **Monotone Sobolev maps** provide the weak/strong closure of $W^{1,p}$-homeomorphisms (Iwaniec-Onninen 2012).
- 2D, $p < 2$: Equality of weak/strong closure, topological characterization of limits (Philippis-Pratelli 2017).

Basic questions to understand about minimizers

Regularity:

- How smooth are minimizers?
- Can there be loss of continuity (when $p < 2$)?

Topology (loss of injectivity):

- 2D, $p \geq 2$: **Monotone Sobolev maps** provide the weak/strong closure of $W^{1,p}$-homeomorphisms (Iwaniec-Onninen 2012).
- 2D, $p < 2$: Equality of weak/strong closure, topological characterization of limits (Philippis-Pratelli 2017).

Symmetry:

- Non-radial minimizers for the p-harmonic energy exist for planar annuli (K-Onninen 2018).
- Sufficient conditions to guarantee a radial minimizer for the bi-Sobolev energy $\int_{\mathbb{A}} |Dh|^p + \int_{\mathbb{A}^*} |Dh^{-1}|^p$ (K-Jussinmäki).

Thank you!