GENERAL TOPOLOGY

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Contents

Introduction

Content of these notes

First and foremost, these notes contain the basic definitions and results needed to start building the theory of general topology.

Some of the lemmas and theorems are accompanied with proofs. These are mainly there for the interested reader who wishes to know the details of why a result holds true. It is not recommended to memorize any of the proofs, rather one might use them as reference material on what type of argumentation is required to prove results in topology.

There are also proofs which are omitted or left as exercises at the end of each section.

There are also a fair number of examples about different topological spaces, maps, and so on, which tie into one of the main goals of these lecture notes: To develop the rigorous reasoning and critical thinking skills of the reader. Intuition is a very strong tool, and indeed one can get quite far in mathematics with a strong sense of intuition even without having to deal with rigorous reasoning. If you've studied Euclidean spaces and metric spaces before, chances are you've got a fairly strong intuition when it comes to things such as limits, continuity, and overall geometrical thinking. The aim of the examples we present here is to break that intuition, since many of the familiar and "reasonable" facts we know to hold in the Euclidean setting stop working in the context of general topological spaces. Once intuition stops being reliable there is no choice but to rely on proving things directly and precisely from the definitions we are given. It is quite like turning off the lights inside the house to learn how to navigate with our other senses than sight. The idea is that later when we turn the lights back on again, we have emerged with new knowledge and skills that will be beneficial in our future studies.

Prerequisites

We assume that the reader is somewhat familiar with topology in metric spaces. Though the theory in these notes is fairly self-contained, we do make some references and comparisons to metric and normed space topology.

Some basic set theory and calculus is also assumed. Countability in particular appears in many different contexts throughout these notes, so we have dedicated a the first section as a rehearsal on these concepts to prepare the reader for later treatment of countable and uncountable sets.

Other sources

In making these notes, we took some inspiration from the following books:

- Jussi Väisälä, *Topologia II*, Limes ry. 2004
- James Munkres, *Topology*, Prentice Hall inc. 2000

We warmly recommend the reader with further interest in topology to take a look at these sources.

Set theory and other preliminaries

Before we get to the main topic of these notes, let us go through some required knowledge in this section.

0.1 Basic notation

In these notes, we denote by

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers.
- $\mathbb{Z}$ the set of integers.
- $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ the set of positive integers.
- $\mathbb{Q}$ the set of rational numbers.
- $\mathbb{R}$ the set of real numbers.
- (a, b) an open interval on the real line, $[a, b]$ a closed interval.
- $\mathbb{R}^n$ the set of n-dimensional Euclidean space.
- $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ the unit circle in the plane (more generally, S^n denotes the n-dimensional ball).

Let A, B be some sets. We also suppose that $\{A_i\}, i \in I$ is some collection of sets, with I being another set (which we usually distinguish by calling it an *index set*). Then

- The **empty set** is denoted by $\emptyset$.
- We denote $A \subset B$ if A is a **subset** of A, meaning that for every $x \in A$, also $x \in B$. This also allows for the case $A = B$. Strict subsets are denoted by $A \subsetneq B$.
- The **union** of A and B is denoted by $A \cup B$ and defined as $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The union of a collection of sets $\{A_i\}$ is defined as $\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$. Or we may write $\bigcup\{A_i : i \in I\}$.
- The **intersection** of A and B is denoted by $A \cap B$ and defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- The intersection of a collection of sets $\{A_i\}$ is defined as $\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$. Or we may write $\bigcap\{A_i : i \in I\}$.
- The **complement** of a subset $A \subset B$ is denoted by $B \setminus A$ and defined as $B \setminus A = \{x \in B : x \notin A\}$. When B is clear from context we simply denote this complement by A^{c}.
- The **product** of A and B is denoted by $A \times B$ and defined as the collection of ordered pairs (a, b) with $a \in A$ and $b \in B$.

We next define the precise set-theoretic meanings of functions and sequences. In practice, it is enough to know that a function $f : X \to Y$ is simply a way to assign to each point $x \in X$ a value $f(x) \in Y$, but we give the precise definitions here for completeness.

A **function** $f : X \to Y$ between sets X, Y is, more precisely, a collection of three sets (X, Y, Z) with $Z \subset X \times Y$. The set X is called the **domain** of f and the set Y is called the **target** of f. For the set Z, we assume that for each $x \in X$, there is exactly one point $z \in Z$ such that $z = (x, y)$ for some $y \in Y$. For each $x \in X$, this particular y is called the **value of** f **at** x and denoted by $y = f(x)$.

- The **image** of a subset $A \subset X$ under f is defined as $f(A) := \{y \in Y : y = f(x) \text{ for some } x \in A\}$.
- The **preimage** of a set $B \subset Y$ is defined as $f^{-1}(B) = \{x \in X : f(x) \in B\}$.
- The **restriction** of a function f to a subset $A \subset X$ is denoted by $f|_A$, and this is defined as a function $f|_A : A \to Y$ by setting $f|_A(x) = f(x)$ for all $x \in A$.

A **sequence** $(x_n)_{n \in \mathbb{Z}_+}$ of points in a space X is, more precisely, a function $x : \mathbb{Z}_+ \to X$, where we use the shorthand notation $x_n := x(n)$. We usually denote a sequence just by (x_n), choosing to omit the part $n \in \mathbb{Z}_+$ as this is always assumed. The notation $(x_n) \subset X$ simply means that $x_n \in X$ for all $n \in \mathbb{Z}_+$ (or in other words, the set obtained as the union of all points $\{x_n\}$ is a subset of X).

0.2 Countable and uncountable sets

In many areas of mathematics, one is mainly concerned with the study of sets with an infinite number of points. But even infinite sets have different orders of magnitude, and perhaps the simplest infinite sets are those which are countably infinite.

Definition 0.1. A set A is **countably infinite** if there exists a bijection

$$f : \mathbb{Z}_+ \to A.$$

A set is **countable** if it is either finite or countably infinite.

In simple terms, the function f gives a way to "count" the elements of the set A in terms of the positive integers: The first element is $f(1)$, the second $f(2)$, and so on.

Example 0.1. The set $\mathbb{Z}$ is countable, as illustrated by the bijection $f : \mathbb{Z}_+ \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} 0 & \text{when } n = 1, \\ n/2 & \text{when } n = 2k \text{ for } k \in \mathbb{Z}_+ \\ -(n-1)/2 & \text{when } n = 2k+1 \text{ for } k \in \mathbb{Z}_+ \end{cases}.$$

Intuitively speaking, this map enumerates the set of integers by covering the set of positive integers by the set of even positive integers, and the set of negative integers by the set of odd positive integers larger than 1.

Example 0.2. The sets $\mathbb{Z}_+ \times \mathbb{Z}_+$ and $\mathbb{Q}$ are countable.

The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ consists of pairs (a, b) of positive integers $a, b \in \mathbb{Z}_+$. While it may sound counter-intuitive that the 'two-dimensional' set $\mathbb{Z}_+ \times \mathbb{Z}_+$ can actually be covered by the 'one-dimensional' set $\mathbb{Z}_+$, this is entirely possible.

We use the following picture to illustrate this process, drawing each point in $\mathbb{Z}_+ \times \mathbb{Z}_+$ at the corresponding point in the plane:

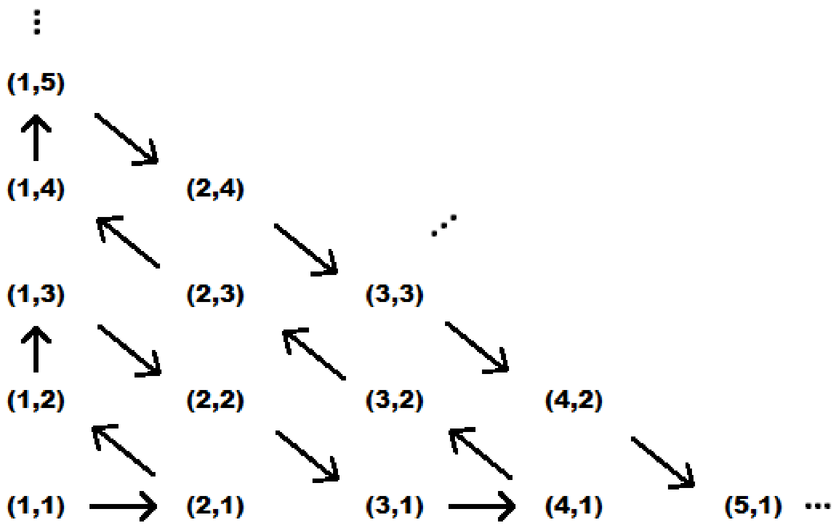


Figure 1: A way to enumerate each point in $\mathbb{Z}_+ \times \mathbb{Z}_+$ by travelling through the points in the path illustrated in the figure, which eventually covers every integer point in the first quadrant.

The easy way to interpret this is that the path shown in the figure gives a map $f : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$, where $f(n)$ denotes the point on the path we are at on the n:th step. This map f ends up being bijective because each point in $\mathbb{Z}_+ \times \mathbb{Z}_+$ is eventually covered by such a path and no point is covered twice. Hence $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable.

The set of rational numbers can be shown to be countable in much the same way, as each rational number $q \in \mathbb{Q}$ may be written as $q = a/b$ for some integers $a, b \in \mathbb{Z}$, $b \neq 0$. While it is possible that a rational number has multiple representations (for example, $1/2 = 2/4$), this representation can be made unique by assuming that a and b have no common factors.

In this way, we can expand on the previous case of $\mathbb{Z}_+ \times \mathbb{Z}_+$ to show that the set of positive rational numbers are countable. In fact, we may represent each positive rational number as $q = a/b$ with $a, b \in \mathbb{Z}_+$, and identify this with the point (a, b) in $\mathbb{Z}_+ \times \mathbb{Z}_+$. Then we can use the previous path to construct a bijection by using the same path as before but skipping each occurrence of a pair (a, b) where a and b already have a common factor. For example, we skip the pair $(4, 2)$ because it represents the same rational number as $(2, 1)$.

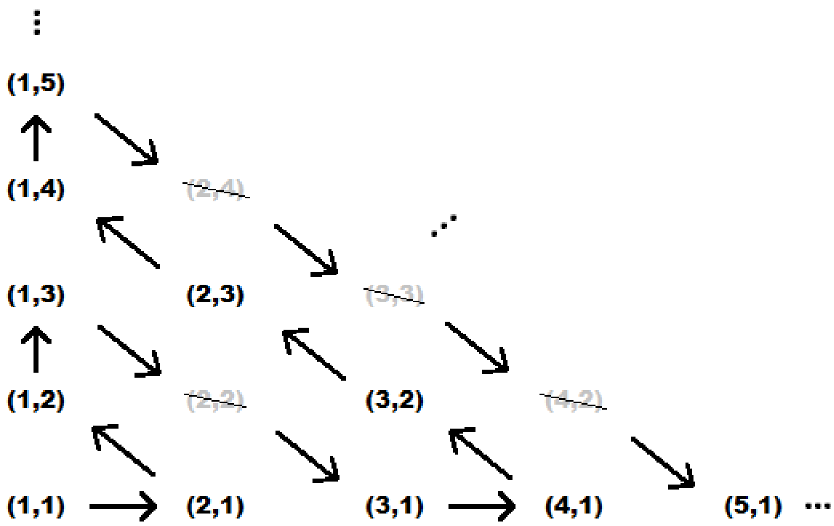


Figure 2: A way to enumerate each positive rational number $q \in \mathbb{Q}$, $q > 0$, by representing them as pairs (a, b) in the first quadrant of the plane.

Of course, this so far only shows that the set of positive rational numbers is countable. We leave it as an exercise to prove that also the whole set $\mathbb{Q}$ is countable.

Sets which are not countable are called **uncountable** or **uncountably infinite** as they are necessarily infinite sets. The real numbers give the most basic example.

Theorem 0.1. *The real numbers are uncountable.*

Proof. We argue by contradiction and suppose that $\mathbb{R}$ is countable. The idea is to use a trick based on the fact that each real number has a decimal representation, which we assume to be known here. Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be the given bijection and write each output of the function f in its decimal representation, for example:

$$\begin{array}{ll} f(1) = 0.88192387\ldots & \qquad f(1) = 0.88192387\ldots \\ f(2) = 4.24998899\ldots & \qquad f(2) = 4.24998899 \\ f(3) = 3.96000000\ldots & \qquad f(3) = 3.96000000 \\ f(4) = 0.33333333\ldots & \qquad f(4) = 0.33333333\ldots \\ \quad\vdots & \qquad\quad\vdots \end{array}$$

Here on the right hand side, we have for each line n highlighted the n:th decimal of the representation of $f(n)$, let's call this digit D_n (so $8, 4, 0, 3, \ldots$ in the example). We then construct a real number x by giving its decimal representation as follows. The decimal representation of x will be $x = 0.d_1d_2d_3d_4\ldots$, where each digit d_n is chosen to be a number different from D_n and between 1 to 8 (in the example, $x = 0.7312\ldots$ works). We claim that there is no $N \in \mathbb{Z}_+$ so that $f(N) = x$.

If such a N would exist, then looking at the N:th decimal place of both numbers $f(N)$ and x, we would find digits D_N and d_N which, by definition of d_N, are different. Moreover, two numbers cannot have the same decimal representation unless one ends with a trailing sequence of 9:s and the other with a trailing sequence of 0:s, for example $1.0000\ldots = 0.9999\ldots$ is the classic case of this. However, none of the digits d_n are 0:s or 9:s by choice, so this gives a proper contradiction. This proves the result. □

Sometimes it is annoying to construct a bijection from a set A to $\mathbb{Z}_+$, but it is otherwise reasonable to argue that A should be countable because it contains "at most as many" points as $\mathbb{Z}_+$. This can be made more precise with the following result.

Theorem 0.2.
- *Suppose that A is a set and that there exists an injection $g : A \to \mathbb{Z}_+$. Then A is countable.*
- *Suppose that A is a set and that there exists a surjection $h : \mathbb{Z}_+ \to A$. Then A is countable.*

We leave the proof as an exercise.

Lastly, we list a few more key results which can be helpful in determining that a given set is countable.

Theorem 0.3. *The following facts hold:*
- *The union of a countable collection of countable sets is countable.*
- *If X, Y are countable, then $X \times Y$ is countable.*
- *A countable set has countably many finite subsets.*

Proof. We give only a sketch of the ideas here and leave the details for the reader.

For the first point, let's suppose that the union is disjoint (the other case is arguably even easier as there are 'less' elements). We define an injective map from this countable union into $\mathbb{Z}_+$ by mapping the elements of the n:th set to the numbers p_n^k, $k = 1, 2, \ldots$, where p_n is the n:th prime number. The main point here is that the sequences $(p_n^k)_{k=1}^{\infty} \subset \mathbb{Z}_+$ are disjoint countably infinite subsets of $\mathbb{Z}_+$, and there are enough of such sequences since there are infinitely many primes.

For the second point, one may repeat the proof of showing that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable or simply find first a bijection $h : A \times B \to \mathbb{Z}_+ \times \mathbb{Z}_+$, and then compose with a bijection from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to $\mathbb{Z}_+$ to obtain a bijection from $A \times B$ to $\mathbb{Z}_+$.

For the last point, one only needs to check that the collection of subsets with exactly N elements is countable, since the total collection of subsets with finite elements is a union of these sets over all N which can then be argued to be countable by the first point. To count the collection of subsets with exactly N elements, one may note that for each M the collection of subsets with N elements which only involve the first M elements is finite (and hence countable). Then applying the first point again over all M gives the result. □

Note, however, that the collection of *all* subsets of a countably infinite set is uncountable and in fact of equal cardinality to $\mathbb{R}$.

0.3 Metrics and norms

We briefly recall what we mean by a **metric** or **norm** on a space.

To start off, we define the concept of a *metric* which is used to quantify the concept of a distance between two points in a space.

Definition 0.2. Let X be some space. A **metric** $d : X \times X \to [0, \infty)$ is a function which satisfies the following three properties.

(**M1**) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(**M2**) $d(x, y) = 0$ if and only if $x = y$.

(**M3**) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A **metric ball** is a subset of X defined by a center x and a radius r:

$$B_d(x, r) := \{y \in X : d(x, y) < r\}.$$

When the space X is a vector space, it is more suitable to use a *norm* to measure the distance between two points as this concept interacts with the scalar multiplication and addition operators defined for vectors in X.

Definition 0.3. Let X be a vector space. A **norm** $||\cdot|| : X \to [0, \infty)$ is a function which satisfies the following three properties.

(**N1**) $||x|| = 0$ if and only if $x = 0$.

(**N2**) $||\lambda x|| = |\lambda|\, ||x||$ for all $\lambda \in \mathbb{R}, x \in X$.

(**N3**) (Triangle inequality) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$.

Every normed space is also a metric space, since a norm $||\cdot||$ defines a metric via the relation $d(x, y) := ||x - y||$.

0.4 Exercises

Exercise 0.1. Show that a subset of a countable set is countable.

Exercise 0.2.

a) Given two countable sets A and B which are disjoint, show that also their union $A \cup B$ is countable by constructing a bijection $f : \mathbb{Z}_+ \to A \cup B$.

b) Use Theorem 0.2 to show that this also holds when A and B are not necessarily disjoint.

Exercise 0.3. Prove Theorem 0.2.

Exercise 0.4. Find a bijection from $\mathcal{P}(\mathbb{N})$ (the set of all subsets of $\mathbb{N}$) and $\mathbb{R}$.

Exercise 0.5. Let A be any set. Show that there is no bijection $f : \mathcal{P}(A) \to A$.

Exercise 0.6. Let S be the subset of real numbers which can be defined using finite amounts of text written in the English alphabet. Prove that S is countable.

Exercise 0.7. Let X be the set of sequences (x_n) of positive integers which are eventually constant, meaning that there is some $N \in \mathbb{Z}_+$ (which may depend on the sequence) such that $x_{n+1} = x_n$ for $n \geq N$. Is X countable?

1 Topological spaces

In this section we introduce the concept of a topological space and detail some of the basic definitions and properties we will use throughout this document.

For a reader familiar with metric spaces, we give some motivation before proceeding further. Recall that a subset U of a metric space is **open** if for every $x \in U$ there is a radius $r > 0$ such that the metric ball $B(x, r)$ is contained in U. A consequence of this definition is that arbitrary unions or finite intersections of open sets are still open sets. These two key properties of open sets in metric spaces are taken as the starting point for the definition of a general topological space. Hence, in topological spaces we discard the need for a metric but keep only the notion of open sets.

1.1 Basic definitions

The basic idea in studying topological spaces is to take some mathematical object, a set, and equip it with a structure called a *topology* which will let us talk about many interesting properties such as convergence and continuity in that set. We will give the basic definition right here, but this is quite an abstract concept and it will take some time until all of its benefits are revealed.

Definition 1.1. A **topology** $\mathcal{T}$ on a set X is a collection of subsets of X that satisfies the following three axioms:

(**A1**) The empty set $\emptyset$ and the whole set X belong to $\mathcal{T}$.

(**A2**) The union of any collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$. More precisely, for any index set I

$$U_i \in \mathcal{T}, \forall i \in I \;\Rightarrow\; \bigcup_{i \in I} U_i \in \mathcal{T}.$$

(**A3**) The intersection of any finite collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$. More precisely, for any natural number n

$$U_1, \ldots, U_n \in \mathcal{T} \;\Rightarrow\; \bigcap_{i=1}^{n} U_i \in \mathcal{T}.$$

Note!

The index set I in **(A2)** can even be an uncountable set!

The sets in $\mathcal{T}$ are referred to as **open sets** in X with respect to the topology $\mathcal{T}$. Hence the unions and finite intersections of open sets are still open sets.

We next make it precise what we mean by a topological space.

Definition 1.2. A **topological space** is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

If the choice of a topology on X is clear from context or doesn't need to be specified, we prefer to omit it from the notation and simply say that "X is a topological space". More precisely, this phrase means that X is a set and there is some topology prescribed on X.

Example 1.1. The **discrete topology** on X is the topology consisting of all subsets of X. In particular, singletons $\{x\}$ are open sets for any $x \in X$.

Example 1.2. Any metric space (X, d) is also a topological space when endowed with the **metric topology** $\mathcal{T}_d$, which is simply defined via the familiar definition of open sets in the metric setting: a set U belongs to $\mathcal{T}_d$ if for every $x \in U$ there is $r > 0$ s.t. the metric ball $B_d(x, r) \subset U$.

Let $(X, \mathcal{T})$ be a topological space. If there is a metric $d : X \times X \to \mathbb{R}$ such that $\mathcal{T} = \mathcal{T}_d$, then we say that $(X, \mathcal{T})$ is **metrizable**.

We make some note of various terminology.

Note!

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same space X, and suppose that $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case we say that $\mathcal{T}_2$ is a **finer** topology than $\mathcal{T}_1$, and $\mathcal{T}_1$ is a **coarser** topology than $\mathcal{T}_2$.

The finest topology in a space is the *discrete topology* of Example 1.1, and the coarsest is the *mini-topology* simply equal to $\mathcal{T}_{mini} = \{\emptyset, X\}$.

Intuitively, one could think of a topology as the collection of rocks and pebbles that make up a space. Breaking these rocks (the open sets) into smaller pieces provides a finer topology on the same space. Indeed, if $\mathcal{T}_1 \subset \mathcal{T}_2$ then each open set $U \in \mathcal{T}_1$ is also an open set in $\mathcal{T}_2$, and there could be some parts of U which are open in $\mathcal{T}_2$ but not in $\mathcal{T}_1$. This analogy is not perfect though, since it is not necessarily true that an open set U in $\mathcal{T}_1$ will be the disjoint union of its 'pieces' in $\mathcal{T}_2$, consider for example $X = \{1, 2\}$ equipped with topologies $\mathcal{T}_1 = \{\emptyset, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{1\}, X\}$. Here the set X is in both topologies and its 'piece' $\{1\}$ is also in $\mathcal{T}_2$, but the other part $\{2\}$ is not in $\mathcal{T}_2$.

Next, we introduce a commonly used shorthand notation called *neighbourhoods*.

Note!

Let $(X, \mathcal{T})$ be a topological space and $U \in \mathcal{T}$ be an open set.

If $x \in X$ is a point such that $x \in U$, we say that U is a **neighbourhood** of x.

If $A \subset X$ is a set such that $A \subset U$, we say that U is a **neighbourhood** of the set A.

Every point x in a topological space has at least one neighbourhood, since the whole space X is a neighbourhood of every point. A neighbourhood does not need to contain multiple points: if x has a neighbourhood that contains only the point x (i.e. $\{x\}$ is open), we say that x is an **isolated point**. In the discrete topology every point is an isolated point.

We make one final remark about the inclusion notation:

Note!

In these notes, inclusion "$\subset$" means **non-strict inclusion**, in other words if $A \subset B$ then we allow that $A = B$. For strict inclusion we would use $A \subsetneq B$.

1.2 Closed sets and related concepts

Throughout the remainder of this section, the symbols $(X, \mathcal{T})$ refer to some topological space X and its topology $\mathcal{T}$.

1.2.1 Closed sets

Note!

When the space we are working with (in this case X) is clear from context, we denote by A^{c} the complement of a subset $A \subset X$, precisely $A^{\mathsf{c}} := X \setminus A$.

We begin by defining closed sets.

Definition 1.3. A set $F \subset X$ is a **closed set** if its complement F^{c} is open.

Closed sets constitute a natural counterpart to open sets. Since taking complements switches unions $\bigcup$ into intersections $\bigcap$ and vice versa, it is direct to show from the definition of open sets that:

- The empty set $\emptyset$ and the whole space X are closed.
- The intersection of any collection of closed sets is closed.

- The union of any finite collection of closed sets is closed.

Likewise, given a collection of subsets on X satisfying the above three properties, we can obtain a topology on X by taking their complements as the open sets. Sometimes this property is used to define a topological space via its closed sets rather than its open sets if it is more appealing to do so instead.

Note!

There can be sets which are both open and closed. The empty set and the whole space are always examples of this. In the discrete topology (Example 1.1), every set is both open and closed.

1.2.2 Interior, exterior, and boundary

We next define the interior, exterior, and boundary of a set.

Definition 1.4. Let $A \subset X$ be a subset of the space X.

- A point $x \in A$ is called an **interior point** if there is a neighbourhood U of x such that $U \subset A$. The collection of all interior points is called the **interior** of A and denoted by $\operatorname{int} A$:

$$\operatorname{int} A = \{x \in A : \exists U \subset A \text{ with } x \in U \text{ and } U \text{ open}\}.$$

- An **exterior point** of A is any interior point of the complement A^{c}. The collection of all exterior points is called the **exterior** of A and denoted by $\operatorname{ext} A$.
- A point $x \in X$ is called a **boundary point** of A if it belongs to neither $\operatorname{int} A$ or $\operatorname{ext} A$. The **boundary** of A is denoted by $\partial A := X \setminus (\operatorname{int} A \cup \operatorname{ext} A)$.

An alternative way to define the interior of A is given by the formula (exercise)

$$\operatorname{int} A = \bigcup \{U : U \subset A \text{ and } U \text{ is open}\}. \tag{1.1}$$

This formula holds because 1) By definition every interior point of A is contained in some member of the union on the right hand side, and 2) Every set U within the union on the right hand side must consist entirely of interior points of A by the definition of an interior point.

The main consequence of (1.1) is that the interior $\operatorname{int} A$ is always an open set (exercise). In a similar vein, the exterior $\operatorname{ext} A$ is also an open set while the boundary ∂A is closed. The exterior is also the same thing as the interior of the complement, so $\operatorname{ext} A = \operatorname{int} A^{\mathsf{c}}$ (exercise).

Example 1.3. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ are a subset with no interior or exterior. Hence the boundary of $\mathbb{Q}$ is equal to the whole real line, $\partial\mathbb{Q} = \mathbb{R}$.

1.2.3 Closure

We next define the closure of a set A.

Definition 1.5. The **closure** of a set A, denoted by $\bar{A}$, is defined as $\bar{A} := \operatorname{int} A \cup \partial A$.

The closure of a set is always a closed set. Indeed, by definition the closure $\bar{A}$ is simply equal to the complement of the exterior $\operatorname{ext} A$ which we determined to be open. Another way to deduce this is via the alternative representation

$$\bar{A} = \bigcap \{F : A \subset F \text{ and } F \text{ is closed}\}, \tag{1.2}$$

which we leave as an exercise. The following lemma is also an exercise.

Lemma 1.1.

- *A is open if and only if $A = \operatorname{int} A$.*
- *A is closed if and only if $A = \bar{A}$.*

We next give perhaps the most useful result about the boundary and the closure of a set. This result gives new characterizations which could also be taken as alternative definitions for the boundary and closure.

Theorem 1.2.

- *The boundary ∂A of a set $A \subset X$ is precisely the set of points $x \in X$ for which every neighbourhood of x intersects both A and A^{c}.*
- *The closure $\bar{A}$ of a set $A \subset X$ is precisely the set of points $x \in X$ for which every neighbourhood of x intersects A.*

Proof. Let's prove the statement about ∂A first. If every neighbourhood of a point x intersects A^{c} then it's not possible that $x \in \operatorname{int} A$ as then x cannot be an interior point. Similarly if every neighbourhood of x intersects A then it's not possible that $x \in \operatorname{ext} A$. Hence if every neighbourhood of x contains points from both A and A^{c} then x has to belong to the boundary ∂A.

Conversely, if $x \in \partial A$ and U is a neighbourhood of x, then U cannot be entirely contained in either A or A^{c} or we would have $x \in \operatorname{int} A$ or $x \in \operatorname{ext} A$ respectively. Hence U must meet both A and A^{c}.

For the second statement about $\bar{A}$, note that every neighbourhood of x intersecting A is equivalent with saying that either: 1) Some of the neighbourhoods are entirely contained in A, in which case $x \in \operatorname{int} A$. Or 2) Every neighbourhood also intersects A^{c}, in which case $x \in \partial A$.

Hence the points x for which every neighbourhood intersects A are exactly the points in the set $\int A \cup \partial A = \bar{A}$. ☐

We next detail some properties of the closure operation, left as exercises.

Theorem 1.3. *Let $A, B \subset X$. Then*

- $\bar{\bar{A}} = \bar{A}$.
- *If $A \subset B$, then $\bar{A} \subset \bar{B}$.*
- *If $A \subset B \subset \bar{A}$, then $\bar{B} = \bar{A}$.*
- $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
- $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

1.2.4 Accumulation points

Accumulation points are another concept deeply related to the notions discussed in the previous subsection.

Definition 1.6. A point $x \in X$ is an **accumulation point** of the set $A \subset X$ if each neighbourhood U of x contains a point in A *other than x itself.*

The set of accumulation points of A is denoted here by $\operatorname{acc} A$. The restriction of U containing points *other than x itself* in the definition is simply a convention which intuitively says that accumulation points should not exist isolated from other points in A.

The notion of accumulation points gives another characterization for the closure of a set via the following formula, left as an exercise:

$$\bar{A} = A \cup \operatorname{acc} A. \tag{1.3}$$

Combining (1.3) with Lemma 1.1 gives an alternative characterization of closed sets:

A set is closed if and only if it contains all of its accumulation points.

Note!

Some textbooks use different terminology for accumulation points. The terms *cluster points* and *limit points* are sometimes used instead. In this document, we make a clear distinction:

- **Accumulation points** are defined as above and are mainly a notion for sets.
- **Limit** is a term reserved for the limit of a sequence, see Section 2.1.
- **Cluster points** is a term reserved for sequences, see Section 2.1.1.

1.3 Relative/subspace topology

One of the basic ways of creating new topological spaces is through subsets of an existing topological space. The canonical way to do this is via the notion of relative topology (also called "subspace topology").

Definition 1.7. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then the **relative topology** on A is a topology on A defined by

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}.$$

It is fairly simple to verify that $\mathcal{T}_A$ indeed satisfies the axioms (A1-3) of a topology as in Definition 1.1. Hence this makes $(A, \mathcal{T}_A)$ a topological space. By default, we will always use relative topology when defining a new topological space as a subset of an existing one.

The term *subspace* is just a fancy word for *subset*, used to highlight the fact that we are considering the subset as a topological space with the relative topology.

Working with relative topology can be a bit tedious, so we collect some results to make things easier:

Theorem 1.4. *Let $A \subset X$ be a subspace. Then the following hold:*

- *If A is an open set in X, then every subset $B \subset A$ which is open in the relative topology on A is also open in X.*
- *If A is a closed set in X, then every subset $B \subset A$ which is closed in the relative topology on A is also closed in X.*
- *If $\mathrm{cl}_A(\cdot)$ denotes the closure operation in the relative topology on A, then for any $B \subset A$ we have that $\mathrm{cl}_A(B) = \bar{B} \cap A$, where $\bar{B}$ is the closure in the topology of X.*

Proofs of these points are given as exercises.

Note!

Especially in familiar spaces, it is easy to get confused by the fact that normally non-open sets can be open in the relative topology of a subspace. For example, $(0, 1]$ is not open in $\mathbb{R}$ but it is open in the relative topology on $[0, 1]$.

Example 1.4. If $A = [0, 1) \cup [2, 3] \cup (4, 5)$ denotes a set consisting of three disjoint intervals in $\mathbb{R}$, then each of the intervals $[0, 1)$, $[2, 3]$, and $(4, 5)$ is both an open and closed set in the relative topology on A (inherited from the usual topology on $\mathbb{R}$).

In topology, sets which are both open and closed are exactly the sets with no boundary (exercise).

1.4 Basis

In metric spaces the existence of open balls often makes dealing with things simpler and more concrete. While there is no direct counterpart to open balls in the setting of general topological spaces, the concept of a *basis* (also called *base*) offers many similar advantages.

Definition 1.8. A subcollection $\mathcal{B} \subset \mathcal{T}$ is called a **basis** for $\mathcal{T}$ if every open set $U \in \mathcal{T}$ may be written as some union of elements in $\mathcal{B}$:

$$U = \bigcup_{i \in I} B_i, \qquad \text{with } B_i \in \mathcal{B} \text{ for all } i \in I.$$

The members $B \in \mathcal{B}$ are called **basis elements**.

Note that by definition:

- Every basis element is an open set.
- The topology $\mathcal{T}$ is trivially a basis for itself.
- A basis doesn't have to itself be a topology of X. For example, the union of two basis elements doesn't have to be a basis element.
- A basis is not necessarily determined uniquely by its topology. A topology may have multiple bases (see Example 1.5).
- A basis $\mathcal{B}$ determines its topology $\mathcal{T}$ uniquely. In fact, the topology $\mathcal{T}$ must equal to the set of all possible unions of basis elements from $\mathcal{B}$. (Note: We also account for the empty set $\emptyset \in \mathcal{T}$ by allowing empty unions where the index set $I = \emptyset$.)

The advantages of considering a basis are very similar to the advantages of using balls in a metric space: All the topological properties of a space X may be described only in terms of the basis elements, but the basis elements themselves may be much simpler and easier to deal with than general open sets in X.

There are a few equivalent conditions for being a basis, and one important one to remember is the following result which we leave as an exercise.

Theorem 1.5. *A subcollection $\mathcal{B} \subset \mathcal{T}$ is a basis for $\mathcal{T}$ if and only if the following condition holds: For every $U \in \mathcal{T}$ and every $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.*

1.4.1 Generating a topology via basis

Another important aspect of bases is that one can define a topology on a space only in terms of a basis, giving a flexible way to create new topological spaces. The following characterization describes exactly when a given collection of sets defines a basis for some topology on X:

Theorem 1.6. *A collection of subsets $\mathcal{B}$ on a set X is the basis for some topology on X if and only if the following conditions hold:*

- *Every $x \in X$ belongs to some $B \in \mathcal{B}$.*
- *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

Proof. The "only if" part of the theorem is fairly simple: If $\mathcal{B}$ is a basis for a topology on X, then the first condition of Theorem 1.6 holds true because X itself is a union of basis elements. The second condition is true because $B_1 \cap B_2$ is an open set as the intersection of two open sets, and thus also a union of basis elements, hence every $x \subset B_1 \cap B_2$ must belong to at least one such basis element.

Let us now prove the "if" part of the theorem. Given a collection of subsets $\mathcal{B}$ satisfying the two conditions, we let $\mathcal{T}$ denote the set of all unions of members of $\mathcal{B}$, including the empty union $\emptyset$. If we can show that $\mathcal{T}$ is a topology then the fact that $\mathcal{B}$ is a basis of $\mathcal{T}$ follows immediately by Definition 1.8.

We verify the axioms of a topology. Axiom (A1) holds because the empty union $\emptyset$ belongs to $\mathcal{T}$, and because the whole space X belongs to $\mathcal{T}$ by the first condition of the theorem. The fact that $\mathcal{T}$ was defined by taking arbitrary unions of elements from $\mathcal{B}$ also guarantees that further unions of members of $\mathcal{T}$ are still members of $\mathcal{T}$, giving (A2).

For the last axiom it is enough to prove that the intersection of two open sets is open, as the general case will follow by induction. Let $U_1, U_2 \in \mathcal{T}$ and we aim to show that $U_1 \cap U_2 \in \mathcal{T}$. For this it is enough to show that any $x \in U_1 \cap U_2$ belongs to some $B_x \in \mathcal{B}$ such that $B_x \subset U_1 \cap U_2$, as then $U_1 \cap U_2$ must also be the union of such sets B_x. Let $x \in U_1 \cap U_2$. Then $x \in U_1$ and because U_1 is the union of elements of $\mathcal{B}$ we must have $x \in B_1 \subset U_1$ for some $B_1 \in \mathcal{B}$. Similarly $x \in B_2 \subset U_2$ for some $B_2 \in \mathcal{B}$. But now $x \in B_1 \cap B_2$ and by the second point in Theorem 1.6 we find that there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subset B_1 \cap B_2 \subset U_1 \cap U_2$. This concludes the proof. □

Example 1.5. Given a metric space X, the collection of all balls $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$ provides a basis for X.

However, this is not the only possible choice. For example, in the plane $\mathbb{R}^2$ another basis is given by open rectangles $(a, b) \times (c, d) \subset \mathbb{R}^2$. This basis has the nice property that the intersection of two basis elements is also a basis element (or empty), unlike for balls where the intersection of two balls is not always a ball.

Example 1.6. A basis for the discrete topology on X is given by singletons $\mathcal{B} = \{\{x\} : x \in X\}$, and this is the smallest possible basis for the discrete topology since every other basis must contain the singletons.

1.4.2 Generating a basis via subbasis

Though we just saw that not every collection of subsets can be used as a basis, we may still ask:

Given a collection $\mathcal{S}$ of subsets of X, what is the coarsest (smallest) topology of X which contains the sets in $\mathcal{S}$?

The idea here is that we want a topology $\mathcal{T}$ on X where every set $S \in \mathcal{S}$ is open, but want to avoid adding as much extra material to $\mathcal{T}$ as possible. The two equivalent ways to find this topology $\mathcal{T}$ are:

- Define $\mathcal{T}$ as the intersection of all topologies of X which contain $\mathcal{S}$ (exercise).
- A constructive method: Define $\mathcal{B}$ as the collection of all finite intersections of elements in $\mathcal{S} \cup \{X\}$. Then $\mathcal{B}$ defines a basis for a topology $\mathcal{T}$ on X (proof below).

The reason we wrote $\mathcal{S} \cup \{X\}$ in the latter condition is to guarantee that the first axiom of a topology is satisfied: It is always required that $X \in \mathcal{T}$, meaning that the whole space needs to be an open set. If the elements of $\mathcal{S}$ do not cover the space X entirely (their union is not equal to X), then we take care of this axiom by adding X first.

The collection $\mathcal{S}$ is called a **subbasis** or **prebasis** of $\mathcal{T}$.

Definition 1.9. A subset $\mathcal{S} \subset \mathcal{T}$ is called a **subbasis** for $\mathcal{T}$ if the set

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{N} S_i \ : \ N \in \mathbb{N}, S_i \in \mathcal{S} \ \forall i \right\}$$

is a basis of $\mathcal{T}$.

If $\mathcal{S}$ is a collection of subsets in X such that its union is the whole space X, then $\mathcal{S}$ is the subbasis of some topology X. In summary, this topology is found by taking the arbitrary unions of all finite intersections of sets in $\mathcal{S}$.

Note!

In some other sources, it is required that a subbasis $\mathcal{S}$ covers the space X (to avoid the step where we needed to add X earlier).

Finally, let us prove that the constructive method we did earlier to define the topology $\mathcal{T}$ via a subbasis $\mathcal{S}$ works.

Proof. Let $\mathcal{B}$ be the collection of all finite intersections of elements in $\mathcal{S} \cup \{X\}$ as before. To check that $\mathcal{B}$ defines a basis, we use Theorem 1.6.

The first point is clearly true since we added the set X. For the second point, let $B_1 = \bigcap_{i=1}^{n} S_i$ for some $S_i \in \mathcal{S} \cup \{X\}$, and let $B_2 = \bigcap_{j=1}^{m} S'_j$ for some $S'_j \in \mathcal{S} \cup \{X\}$. Now $B_1 \cap B_2 = \bigcap_{i=1}^{n} S_i \cap \bigcap_{j=1}^{m} S'_j$, which is still a finite intersection ($n + m$ sets total) of sets in $\mathcal{S} \cup \{X\}$, so $B_1 \cap B_2$ is still a basis element itself. So the second point is trivial, because then any $x \in B_1 \cap B_2$ we may simply choose $B_3 = B_1 \cap B_2$ as the basis element containing x here. □

1.5 Exercises

In these exercises, X, Y are always topological spaces unless otherwise stated.

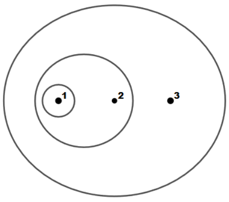
Exercise 1.1. Let $A, B \subset X$ with A open and B closed. Show that $A \setminus B$ is open.

Exercise 1.2. Find a collection of closed sets in $\mathbb{R}^2$ whose union is not closed.

Exercise 1.3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology. What about $\mathcal{T}_1 \cup \mathcal{T}_2$?

Exercise 1.4. Let $X = \mathbb{R}$ and let $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. Prove that $\mathcal{T}$ is a topology.

Exercise 1.5. Let $X = \{1, 2, 3\}$. We have drawn one topology $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, X\}$ for X in the attached picture:



How many topologies of X exist in total?

Exercise 1.6. Let $X = \mathbb{R}$ and $\mathcal{T}' = \{\mathbb{R}\} \cup \{A \subset \mathbb{R} : \mathbb{R} \setminus A \text{ is infinite.}\}$. Is $\mathcal{T}'$ a topology on $\mathbb{R}$?

Exercise 1.7. Let $X = \mathbb{R}$ and $\mathcal{T}_{co} = \{\mathbb{R}\} \cup \{A \subset \mathbb{R} : \mathbb{R} \setminus A \text{ is countable.}\}$. Is $\mathcal{T}_{co}$ a topology on $\mathbb{R}$?

Exercise 1.8. Let $A \subset B \subset X$. Suppose that B is closed and A is a closed set in B (with the relative topology). Show that A is closed in X.

Exercise 1.9. Find the interior and boundary of the following sets in the standard topology of $\mathbb{R}^2$:

(1) $\{(x, y) : y = 0\}$.

(2) $\{(x, y) : x > 0 \text{ and } y \neq 0\}$.

(3) $\{(x, y) : x \in \mathbb{Q}\}$.

(4) $\{(x, y) : 0 < x^2 + y^2 \leq 1\}$.

(5) $\{(x, y) : x, y \in \mathbb{Q}, x > y\}$.

Exercise 1.10. Prove that $\bar{A} = \bigcap \{F : A \subset F \text{ and } F \text{ is closed}\}$.

Exercise 1.11. Determine the closure of the set $\{0\}$ in the topology of Exercise 1.4.

Exercise 1.12. If U is open, does it hold that $U = \operatorname{int}(\bar{U})$?

Exercise 1.13. Show that U is open if and only if $\partial U = \bar{U} \setminus U$.

Exercise 1.14. Prove Theorem 1.3.

Exercise 1.15. Provide an example where $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Exercise 1.16. Let $A_i, i \in I$ be subsets of X indexed by some set I. Prove that $\bigcup_{i \in I} \overline{A_i} \subset \overline{\bigcup_{i \in I} A_i}$. Does equality hold?

Exercise 1.17. Let $A \subset X$. Prove that

- $\partial\partial A \subset \partial A$.
- $\partial\partial A = \partial A$ if A is closed.
- $\partial\partial\partial A = \partial\partial A$.

Exercise 1.18. Prove that $\bar{A} = A \cup \operatorname{acc} A$.

Exercise 1.19. Prove Theorem 1.5.

Exercise 1.20. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Prove that $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if the following holds:

For each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Exercise 1.21. Show that $\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{Q}\}$ provides a basis for the standard topology on $\mathbb{R}$.

Exercise 1.22. Let $A, B \subset X$ be two subsets. What is the coarsest topology on X which contains both the sets A and B?

Exercise 1.23. Given a collection $\mathcal{S}$ of subsets of X. Let

$$\mathcal{T} = \bigcap\{\mathcal{T}' : \mathcal{S} \subset \mathcal{T}' \text{ and } \mathcal{T}' \text{ is a topology of } X\}.$$

Check that this defines a topology $\mathcal{T}$ on X.

Exercise 1.24. Suppose that X is an infinite set with a topology in which every infinite set is open. Show that X has the discrete topology.

Exercise 1.25. Consider the two operations $A \mapsto \bar{A}$ and $A \mapsto A^{\mathsf{c}}$, defined for subsets $A \subset X$. Prove that starting from a set A and using these two operations, one can form no more than 14 distinct sets. Find an example of a subset of $\mathbb{R}$ for which the maximum 14 different sets are attained.

2 Convergence and continuity

Throughout this section, the symbols $(X, \mathcal{T})$ refer to some topological space X and its topology $\mathcal{T}$.

2.1 Sequential convergence and Hausdorff spaces

Throughout this subsection $(x_n) \subset X$ denotes some sequence of points. From calculus we already understand the importance of studying the limiting behaviour of different sequences. The theme here will be to broaden these ideas to general topological spaces. As a word of warning though, we will soon find out that sequences aren't as useful of a tool to work with in general topology as they are in Euclidean and metric spaces, and we will devote some time to discuss their shortcomings.

We start by defining an analogue of sequential limits in topological spaces.

Definition 2.1. A sequence (x_n) **converges** to a point $x \in X$ if, for every neighbourhood U of x, there exists a number $N_U \in \mathbb{Z}_+$ such that $x_n \in U$ for all $n \geq N_U$.

Here x is called a **limit** or **limit point** of the sequence (x_n), this is also denoted by $x_n \to x$.

In the case of Euclidean topology on $\mathbb{R}^n$ (or in general metric spaces), we are accustomed to the fact that a sequence can have at most one limit. This is however not necessarily the case in general topological spaces (which is why we avoid the use of the notation $\lim_{n\to\infty} x_n$ here).

Example 2.1. Let $X = \{-1, 0, 1\}$ and $\mathcal{T} = \{\emptyset, \{-1, 1\}, X\}$. Then the alternating sequence $x_n = (-1)^n$ has two limit points 1 and -1.

In the above example the main issue is that the two points 1 and -1 share all of their neighbourhoods. In most practical applications of topology this type of situation does not occur, which is why there are a fair number of further restrictions usually made to avoid exactly this sort of thing. One of the most commonly used restrictions is given by the notion of Hausdorff spaces:

Definition 2.2 (Hausdorff Spaces)**.** A topological space X is a **Hausdorff space** or simply **Hausdorff** if the following holds: For every pair of distinct points $x, y \in X$, $x \neq y$, there exist neighbourhoods U of x and V of y such that $U \cap V = \emptyset$.

The condition of X being a Hausdorff space basically says that points in X may be separated from each other by disjoint neighbourhoods. Hausdorff spaces are also sometimes called T_2**-spaces**. We next distinguish some main properties of Hausdorff spaces.

Lemma 2.1. *Let X be a Hausdorff space. Then a sequence $(x_n) \subset X$ may converge to at most one point in X.*

Proof. Let (x_n) be a sequence converging to two distinct points x, y. Then x and y have disjoint neighbourhoods U and V respectively, and by the definition of sequential convergence we find N_U such that $x_n \in U$ for all $n \geq N_U$. Since then $x_n \notin V$ for all $n \geq N_U$, we find a contradiction with the fact that (x_n) converges to y. □

Another nice property to keep in mind is that just like in $\mathbb{R}^n$, singletons are closed sets in Hausdorff spaces.

Lemma 2.2. *Let X be a Hausdorff space. Then singletons $\{x\}$ are closed sets for each $x \in X$.*

Proof. For the first point, note that any point $y \neq x$ in the complement of a singleton $\{x\}$ has a neighbourhood V such that $x \notin V$. Therefore the complement of $\{x\}$ is open and hence $\{x\}$ is closed.

The careful reader may have noticed that we only applied one side of the Hausdorff property here and did not need to use the fact that x has a neighbourhood U separate from V. Indeed, the first point is also valid under a weaker assumption of:

$$\text{For each } x, y \in X, x \neq y,\ y \text{ has a neighbourhood } V \text{ such that } x \notin V.$$

Spaces satisfying this property are called T_1**-spaces**, and these are exactly the spaces where singletons are closed (exercise). The T_1- and T_2-properties are a part of more general *separation axioms* which we study more later in Section 4.1. □

2.1.1 Cluster points and first-countability

Let us return to the discussion about sequences. First, we state the following relationship with the closure.

Lemma 2.3. *Let $A \subset X$ and suppose that $(x_n) \subset A$ is a sequence converging to a point $a \in X$. Then $a \in \bar{A}$.*

In metric spaces a converse version of this result also holds, namely that every point $a \in \bar{A}$ may be approached with a sequence of points in A. But in general topological spaces the assumption that $a \in \bar{A}$ does not necessarily guarantee that there is a sequence in A which converges to a. This is one of the indications that sequences are not an effective tool to characterize various topological properties, and the following example further illustrates this.

Example 2.2. The **co-countable topology** on a space X is defined as the topology in which a set U is open if and only if it is empty or its complement U^{c} is countable. One may verify that this indeed is a topology on X.

In the co-countable topology, a sequence (x_n) can only converge to a limit x if there exists a $N \in \mathbb{Z}_+$ such that $x_n = x$ for all $n \geq N$, i.e. the sequence is eventually the constant x. This is because the subsequence (x_{n_k}) of elements which are not equal to x must be countable, and therefore as a set its complement U is open and contains x. By the definition of sequential convergence this means that there is $N \in \mathbb{Z}_+$ such that $x_n \in U$ for all $n \geq N$, but in this case $x_n = x$ for all $n \geq N$ as all the other possible elements of the sequence belong to the complement of U.

Let us now equip the real numbers $\mathbb{R}$ with the co-countable topology. Let also $A = \mathbb{R} \setminus \{0\}$. Note that any open set in this topology must have an infinite (even uncountable) number of points, in particular any open set containing 0 must contain another point and therefore intersect A. Thus $0 \in \bar{A}$. But the only sequences which converge to 0 eventually contain only 0, and therefore no such sequence can be found in A.

We continue our investigation of sequences by introducing another concept, called *cluster points*.

Definition 2.3. A point a is **cluster point** for a sequence (x_n) if every neighbourhood U of a contains infinitely many points in this sequence, i.e. $x_n \in U$ for infinitely many n.

The notion of cluster points is a better way of capturing that a sequence "hangs around" a certain point a fairly often. Indeed, it says that whatever small neighbourhood of a we choose, the sequence will end up visiting it infinitely often.

Note!

Cluster points of a sequence are not to be confused with the accumulation points of a set we defined in Definition 1.6. The usage of the words "limit point", "cluster point" and "accumulation point" is a bit inconsistent throughout literature, but we use these words to mean entirely different things in these notes.

It is clear that if a sequence (x_n) has a subsequence converging to a point a, then a is also a cluster point of (x_n). But remarkably there also exists a cases where a space can have a sequence that has a cluster point but no converging subsequence. This example is quite difficult to construct but we state it here regardless.

Example 2.3. There is a Hausdorff space with a sequence (x_n) and a point a such that a is a cluster point of (x_n) but no subsequence of (x_n) converges to a. See Example A.1 in the appendix for details.

The main reason why the above example exists is because in a general topological space X the set of neighbourhoods of a point $a \in X$ can be quite wild. This issue can be fixed if every point in X has a *countable neighbourhood basis*. We distinguish these spaces as *first-countable spaces*:

> **Definition 2.4.** A topological space X is **first-countable** if for every point $x \in X$, there exists a sequence (U_n) of neighbourhoods of x such that for every neighbourhood U of x, there is $n \in \mathbb{Z}_+$ such that $U_n \subset U$.

Metric spaces are a prime example of this, since the balls $B(x, \frac{1}{n})$ provide a countable neighbourhood basis for every point x. The special part about first-countable spaces is that their topological properties may be characterized entirely in terms of converging sequences.

> **Note!**
>
> The sequence (U_n) in this definition can be taken to be decreasing, i.e. $U_{n+1} \subset U_n$ for all n. This can be shown either by considering intersections $V_n = \bigcap_{k=1}^{n} U_k$ or taking an appropriate subsequence of the (U_n).

In particular, this property makes it much easier to find sequential limits as shown by the following theorem.

> **Theorem 2.4.** *Let X be first-countable. Then if a sequence $(x_n) \subset X$ has a cluster point a, there is a subsequence converging to a.*
>
> *Also, if $A \subset X$ and $a \in \bar{A}$, then there is a sequence in A converging to a.*

Proof. Take the countable collection (U_n) of neighbourhoods given by first-countability, and we may assume that the sequence is decreasing. Since a is a cluster point, each U_n contains infinitely many points in the sequence (x_n). We now define a subsequence (x_{n_k}) inductively.

We pick the first point x_{n_1} as any member of (x_n) inside U_1. Assume now that the first k elements of the subsequence have been chosen and that $x_{n_j} \in U_j$ for all $j \leq k$. We then use the fact that U_{k+1} contains infinitely many members of the original sequence (x_n) to pick an index $n_{k+1} > n_k$ so that $x_{n_{k+1}} \in U_k$. This defines the subsequence and we have the property that each U_k contains x_{n_k} for all k. Furthermore, since $U_j \subset U_k$ for $j > k$, we also have $x_{n_j} \in U_k$ for such j, k.

The subsequence converges to a, since any neighbourhood U of A contains some U_k and therefore also contains all the points x_{n_j} for $j \geq k$. This proves that (x_{n_k}) converges to a.

We leave the second part as an exercise. □

A typical way to overcome the issue of not being able to describe topological concepts in terms of sequential convergence is by broadening the concept of sequences to **nets** or **filters**. In these notes we omit the discussion on nets and filters, but a reader looking to take an even deeper dive into topology should take a look at these concepts from other sources and textbooks.

2.2 Continuous functions

Throughout this subsection $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ denote two topological spaces.

Our first goal will be to explore what it means for a function $f : X \to Y$ between topological spaces to be continuous. We give the definition first and then ponder afterwards how to interpret it.

> **Definition 2.5.** A function $f : X \to Y$ is called **continuous** if whenever $V \subset Y$ is an open set, then the preimage $f^{-1}(V)$ is an open set in X.

This definition might seem a bit odd because it is so different from the usual definition of continuity for functions in Euclidean spaces. Moreover, continuity in $\mathbb{R}^n$ is usually defined pointwise at each point $x_0 \in \mathbb{R}^n$. Intuitively, continuity at a point x_0 should mean the following thing:

A function f is continuous at a point x_0 if whenever x is close to x_0, then $f(x)$ is also close to $f(x_0)$.

There is also an analogue of the pointwise definition of continuous functions between topological spaces. Since there is no metric to tell us the distance between two points, we need to describe the notion of "being close" via arbitrary open sets instead:

> **Definition 2.6.** A function $f : X \to Y$ is **continuous at a point** $x_0 \in X$ if, for every neighbourhood V of $f(x_0)$, there is a neighbourhood U of x_0 such that $f(U) \subset V$.

If a function is not continuous at a point x_0, we say that it is **discontinuous** at x_0.

We next prove a result which connects the above two definitions of continuity in topological spaces, and provides some further equivalent conditions for continuity.

> **Theorem 2.5.** *Let $f : X \to Y$. The following conditions are equivalent.*
>
> *(1) f is continuous.*
>
> *(2) f is continuous at every point $x_0 \in X$.*
>
> *(3) $f^{-1}(F)$ is closed in X for any closed set $F \subset Y$.*
>
> *(4) $f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$.*
>
> *(5) If $\mathcal{B}$ is a basis of the topology of Y, then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.*

Proof. We prove that (1) and (2) are equivalent here, and leave the rest as exercises. It is quite easy to see that (1) implies (2): If $f : X \to Y$ is continuous, then we verify that it is continuous at each $x_0 \in X$ by proving the statement of Definition 2.6. For any neighbourhood V of $f(x_0)$ we can now simply select $U = f^{-1}(V)$ since U is an open set by Definition 2.5 and hence is a neighbourhood of x_0. Moreover, $f(U) = V$ so we are done.

To prove that (2) implies (1) we argue as follows. Let $V \subset Y$ be any open set, and let $x_0 \in f^{-1}(V)$ be an arbitrary point. Then continuity at x_0 implies that there is a neighbourhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset V$, which means that also $U_{x_0} \subset f^{-1}(V)$. This also means that

$$f^{-1}(V) = \bigcup_{x_0 \in f^{-1}(V)} U_{x_0},$$

since the union certainly contains every point in $f^{-1}(V)$ but cannot contain any point outside of it. Hence $f^{-1}(V)$ is an open set since it is a union of open sets in X. □

We next give some basic properties of continuous functions in the form of the following theorem.

Theorem 2.6.

(1) Constant functions are continuous, meaning that if $f(x) = y_0$ for all $x \in X$ and some $y_0 \in Y$, then f is continuous.

(2) The identity function $\mathrm{id}_X : X \to X$ given by $\mathrm{id}_X(x) = x$ for all $x \in X$ is continuous. Note that the topology of X is assumed to be the same on both sides here, otherwise the situation may change.

(3) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f : X \to Z$ is also continuous.

(4) If $f : X \to Y$ is continuous and $A \subset X$ is a subset, then the restriction $f|_A : A \to Y$ is continuous.

Moreover, continuity of a function $f : X \to Y$ is a property which needs to be only checked *locally*, illustrated by the following result.

Theorem 2.7. *Suppose that $f : X \to Y$ is a function such that for each point $x \in X$, there is a neighbourhood U of x such that $f|_U$ is continuous. Then f is continuous.*

In fact, the notion of a *local property* is a powerful concept which appears in many other contexts as well. The terminology is slightly non-precise here, but a property of a topological space is said to hold *locally* if it holds in a neighbourhood of every point in the space. Many topological properties (such as the continuity of a function) are enough to be checked locally, but some give rise to completely new concepts as well (such as *local compactness* or *local connectivity*).

2.2.1 Sequential continuity

The first time we learn about continuity in our life is usually in the context of real functions $f : \mathbb{R} \to \mathbb{R}$, and continuity is often defined as the property of 'if $x_n \to x$, then $f(x_n) \to f(x)$'. In topological spaces, we distinguish this as its separate notion.

Definition 2.7. A function $f : X \to Y$ is **sequentially continuous** if for every sequence $(x_n) \subset X$ converging to a limit x, the sequence $(f(x_n)) \subset Y$ converges to $f(x)$.

The main point here is that a function which is continuous is always sequentially continuous (exercise), but the converse **does not hold in general**. We illustrate this via the following example.

Example 2.4. Let us return to the co-countable topology $\mathcal{T}_{co}$ on a space X as defined in Example 2.2.

Notice first that if X is equipped with the co-countable topology, then any function $f : X \to Y$ to any other space is sequentially continuous. This is because if $x_n \to x$ in the space X, then eventually $x_n = x$ for all large n. Thus also $f(x_n) = f(x)$ for large n, and therefore $f(x_n) \to f(x)$ as any sequence which is eventually constant converges in every possible topology on Y.

However, consider now two different topologies on the real line. We let $(\mathbb{R}, \mathcal{T}_{co})$ be the first space, with $\mathcal{T}_{co}$ denoting the co-countable topology, and let the second space be $(\mathbb{R}, \mathcal{T})$ with $\mathcal{T}$ being the usual metric topology on $\mathbb{R}$. Then the identity function $\mathrm{id}_{\mathbb{R}} : (\mathbb{R}, \mathcal{T}_{co}) \to (\mathbb{R}, \mathcal{T})$ is **sequentially continuous, but it is not continuous** because the preimage of the open interval $(0, 1)$ is simply $(0, 1)$, which is not open in the co-countable topology on $\mathbb{R}$.

As a consequence of such examples, sequential continuity is generally a less useful notion than continuity. However, as we know from $\mathbb{R}^n$ there are also spaces in which these concepts are equivalent. The main requirement here is again the notion of first-countability which we encountered earlier.

Theorem 2.8. *If X is first-countable, then any sequentially continuous function $f : X \to Y$ is continuous.*

There are also other examples of spaces where the conclusion of the above theorem holds, and such spaces are called **sequential spaces**.

2.2.2 Open and closed maps

For continuous maps, the preimage of an open set is always open. But the image of an open set under a continuous map does not need to be open, consider for example $f(x) = x^2$ which maps the open interval $(-1, 1)$ into a non-open interval $[0, 1)$. Maps which do satisfy this property are called *open mappings*.

> **Definition 2.8.** A map $f : X \to Y$ is **open** if $f(A)$ is open for every open $A \subset X$.
>
> A map $f : X \to Y$ is **closed** if $f(A)$ is closed for every closed $A \subset X$.

As we will see in the exercises, maps which are open need not be closed and vice versa.

2.3 Homeomorphisms

Being a homeomorphism is a special property reserved for certain bijections $f : X \to Y$. Recall that every bijection f has an inverse map $f^{-1} : Y \to X$ which is also a bijection.

> **Definition 2.9.** A bijection $f : X \to Y$ is a homeomorphism if both f and f^{-1} are continuous.

Homeomorphisms are especially useful because they give a one-to-one correspondence between open sets in the topologies of their domain and target. Since every topological property of a space depends purely on the properties of its open sets, homeomorphisms preserve all the topological properties between spaces. Hence from a topologists point of view, if two spaces are homeomorphic they are the exact same space.

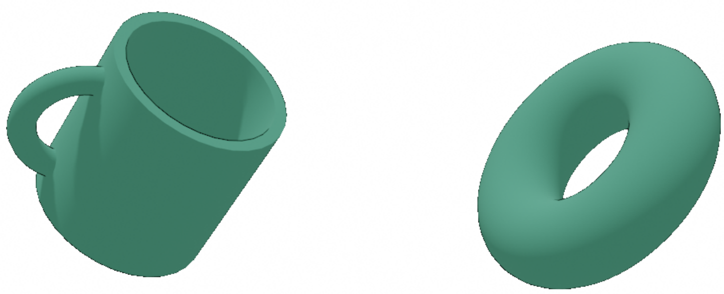


Figure 3: A coffee mug and a donut may look different, but they are homeomorphic and therefore topologically the same object. One way to imagine this is by fashioning a balloon the shape of a coffee mug and then blowing enough air into it until the only detail which remains is the hole from the handle.

Homeomorphisms are open mappings because the image of a set under f is the same as the preimage under its inverse (which is assumed to be continuous). Any continuous open bijection is also a homeomorphism, and the property of being open here is necessary as shown by the following example.

Example 2.5. The set $\mathbb{R}$ is homeomorphic with its subset $(0, 1)$. One homeomorphism between these two subsets may be given by $f(x) = (\pi/2 + \arctan(x))/\pi$ for example.

Example 2.6. A continuous bijection does not need to be a homeomorphism. For example, the function $f : [0, 1) \to S^1$ given by

$$f(t) = (\cos(2\pi t), \sin(2\pi t)),$$

which wraps the half-open unit interval around the circle, is clearly continuous and bijective yet the inverse map fails to be continuous at the point $(1, 0) \in S^1$.

2.3.1 Embeddings

Another special case of homeomorphisms are *embeddings*. Let $f : X \to Y$ be an injection, and let $Z = f(X)$ denote its image set which may not be the whole space Y. Furthermore, let $f' : X \to Z$ denote the function obtained by restricting the range of f to Z instead of Y. Hence f' becomes a bijection of X onto Z. If f' happens to be a homeomorphism as well, we say that the original map f was an **embedding** of X in Y. Hence, f provides a way to represent the topological space X as a subset Z of Y.

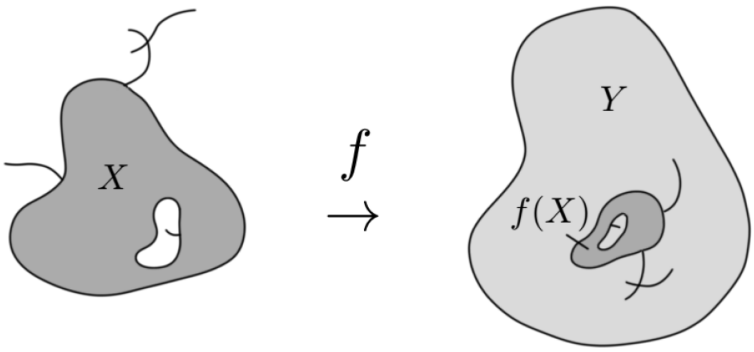


Figure 4: An **embedding** sends the space X onto a homeomorphic copy of it inside the target space Y.

2.4 Spaces of functions

We take a brief detour to illustrate some important examples of topologies one can define on spaces containing functions as elements.

Let us denote by $\mathcal{F}(X, Y)$ the space of functions $f : X \to Y$ between two sets X and Y, which are assumed to be topological spaces here. We will later explore the natural identification of this space with the product space Y^X, but for now we use $\mathcal{F}(X, Y)$ to be more explicit.

2.4.1 The topology of pointwise convergence

We start by making the notion of pointwise convergence topologically precise.

Definition 2.10. For each point $x \in X$ and each open set $U \subset Y$, we define

$$S(x, U) = \{f \in \mathcal{F}(X, Y) : f(x) \in U\}.$$

We then generate a topology on $\mathcal{F}(X, Y)$ using these sets as a subbasis. This is called the **topology of pointwise convergence** and denoted here by $\mathcal{T}_{pw}$.

The name of this topology comes from the following natural property. A sequence $(f_n) \subset \mathcal{F}(X, Y)$ converges to a limit f in the topology $\mathcal{T}_{pw}$ if and only if the following holds:

$$\text{For every } x \in X, \text{ the sequence } (f_n(x)) \subset Y \text{ converges to } f(x). \tag{2.1}$$

Proof. Let $(f_n) \subset \mathcal{F}(X, Y)$ be a sequence and $f \in \mathcal{F}(X, Y)$ be some function.

Let us first assume that (2.1) holds. Let now $V \in \mathcal{T}_{pw}$ be a neighbourhood of f. We must show that $f_n \in V$ for sufficiently large n. Let B be a basis element such that $f \in B \subset V$. Since every basis element is a finite intersection of subbasis elements, write

$$B = \bigcap_{i=1}^{K} S(x_i, U_i)$$

for some finite collection of subbasis elements $S(x_i, U_i)$. It is enough to show that for each $i = 1, \dots, K$, there is N_i such that $f_n \subset S(x_i, U_i)$ for $n \geq N_i$, as then we may choose $N = \max_i N_i$ and obtain that for $n \geq N$ we have $f_n \in S(x_i, B_i)$ for all $i = 1, \ldots, K$ and hence also $f_n \in B$. However, this is precisely what (2.1) says: Given any point $x_i \in X$ and neighbourhood U_i of $f(x_i)$, property (2.1) says that there is N_i such that $f_n(x_i) \in U_i$ for $n \geq N_i$ and hence $f_n \in S(x_i, U_i)$. Hence $f_n \to f$ in $\mathcal{T}_{pw}$.

Conversely, if $f_n \to f$ in $\mathcal{T}_{pw}$ we can simply argue as follows. Each subbasis element $S(x, U)$ is by itself open, so if $f \in S(x, U)$ then $f_n \in S(x, U)$ for sufficiently large n. This directly gives (2.1). □

2.4.2 The topology of uniform convergence

When the target space of our functions is the real line (or any metric space), it is typical to also consider the notion of uniform convergence. This is defined simply as follows.

Definition 2.11. Given $f : X \to \mathbb{R}$ and $r > 0$, we let

$$B(f, r) = \left\{ g \in \mathcal{F}(X, \mathbb{R}) : \sup_{x \in X} |g(x) - f(x)| < r \right\}.$$

The sets $B(f, r)$ define a basis for a topology on $\mathcal{F}(X, \mathbb{R})$, called the **topology of uniform convergence** and denoted here by $\mathcal{T}_{unif}$.

Note that this is almost the same as defining $\mathcal{T}_{unif}$ as the topology generated by the metric $d(f, g) = \sup_{x \in X} |g(x) - f(x)|$ (where the $B(f, r)$ are metric balls). The only difference is that we make no claims above that the quantity $d(f, g)$ is well-defined (not infinite) for all $f, g \in \mathcal{F}(X, \mathbb{R})$. If we further restrict to, say, bounded functions on X ($\sup_{x \in X} |f(x)| < \infty$), then there is indeed no problem in defining such a metric. The space even becomes a normed space with the sup-norm

$$||f||_\infty = \sup_{x \in X} |f(x)|.$$

The topology of uniform convergence is finer than the topology of pointwise convergence.

Uniform convergence is used in many contexts, in particular to construct new continuous functions via the *uniform limit theorem.*

Theorem 2.9 (Uniform Limit Theorem)**.** *Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, and suppose that there is a function $f : X \to \mathbb{R}$ such that*

$$\lim_{n \to \infty} \sup_{x \in X} |f(x) - f_n(x)| \; = \; 0.$$

Then f is continuous.

This result holds even if the target space $\mathbb{R}$ is replaced by a general metric space, and the proof is the same.

Proof. Let $x \in X$ and let $\epsilon > 0$. We show that there is a neighbourhood U of x such that $f(U) \subset (f(x) - \epsilon, f(x) + \epsilon)$.

First, pick N large enough so that $\sup_{x \in X} |f(x) - f_N(x)| \leq \epsilon/2$. Then we use the continuity of f_N to choose a neighbourhood U of x such that $|f_N(x') - f_N(x)| < \epsilon/2$ for $x' \in U$. Thus

$$|f(x') - f(x)| \leq |f(x') - f_N(x')| + |f_N(x') - f_N(x)| < \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{for } x' \in U.$$

This proves the claim. □

2.5 Inducing and coinducing a topology

The concept of continuity also gives us new ways to construct topologies on spaces. Given a function $f : X \to Y$ (or a collection of multiple functions), we can ask two questions

- Given a topology $\mathcal{T}'$ on Y, what is the coarsest topology on X in which $f : X \to Y$ is continuous?
- Given a topology $\mathcal{T}$ on X, what is the finest topology on Y in which $f : X \to Y$ is continuous?

In essence, we want to give optimal assumptions under which a function (or a collection of them) can be made continuous between two spaces.

To make a single map $f : X \to Y$ continuous it is of course required that the preimage of any open set in Y is open in X. Hence for the first question, for every open set $U \in \mathcal{T}'$ we need to at least include the set $f^{-1}(U)$ in our topology on X. In fact, the collection of these sets, namely $\mathcal{T} = \{f^{-1}(U) : U \subset Y \text{ is open}\}$, already happens to define a topology on X which is precisely the topology we are looking for.

The question becomes a bit more nontrivial when we consider a whole collection of functions $f_i : X \to Y_i$ to possibly different topological spaces Y_i with topologies $\mathcal{T}_i$. Naturally, we still must include in our topology the preimage of any open set $U_i \subset Y_i$ under the appropriate map f_i, so the collection

$$S = \{f_i^{-1}(U) : i \in I \text{ and } U \in \mathcal{T}_i\}$$

should at least be a subset of our topology on X. In general the set S is not necessarily a topology by itself, but it can be used as a subbasis (see Section 1.4.2) to generate a topology on X called the *induced topology*.

> **Definition 2.12.** Let X be a set, I be an index set, and $f_i : X \to Y_i$ be a collection of functions into topological spaces $(Y_i, \mathcal{T}_i)$. Then the topology on X generated by the subbasis of sets
> $$S = \{f_i^{-1}(U) : i \in I \text{ and } U \in \mathcal{T}_i\}$$
> is called the topology **induced** by the collection $\{f_i : i \in I\}$.

Based on the above discussion one may verify that each f_i is indeed continuous from the induced topology on X into Y_i, and that the topology induced by the f_i's is indeed the coarsest topology with this property (exercise).

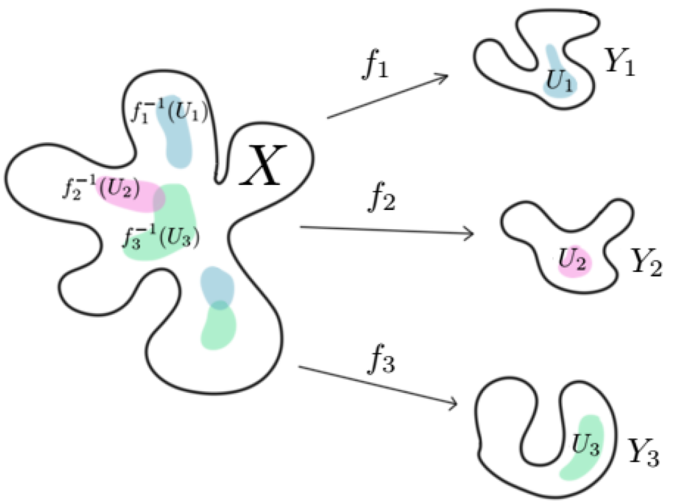


Figure 5: Inducing a topology on X by using the preimages of open sets under a collection of maps $f_i : X \to Y_i$.

We next cover a few key properties of the induced topology. Sequential convergence in the induced topology is simple to characterize:

> **Theorem 2.10.** *Let the topology on X be induced by a family of maps $f_i : X \to Y_i, i \in I$. Then a sequence $(x_n) \subset X$ converges to a limit x if and only if $f_i(x_n) \to f_i(x)$ for all $i \in I$.*

The following result provides both a way to check when a map is continuous into X (with the induced topology) and gives an alternate characterization for the induced topology.

Theorem 2.11. *Let Z be a topological space and $g : Z \to X$. If X has the topology induced by a family $f_i : X \to Y_i$, then g is continuous if and only if $f_i \circ g : Z \to Y_i$ is continuous for all indices i.*

Conversely, if X has a topology $\mathcal{T}$ which has the following property: A map $g : Z \to (X, \mathcal{T})$ is continuous if and only if $f_i \circ g : Z \to Y_i$ is continous for all i. Then $\mathcal{T}$ is the topology induced by the maps f_i.

Both of the above results are given as exercises.

Let's now think about the other direction of finding the finest topology on Y under which a collection of maps $f_i : X_i \to Y$, $i \in I$ is continuous, assuming topologies $\mathcal{T}_i$ on X_i are given. Naturally, we are only allowed to add those subsets $U \subset Y$ in our topology whose preimages $f_i^{-1}(U)$ are open sets in each X_i. It turns out this already defines a topology on Y.

Definition 2.13. Given a collection of functions $f_i : X \to Y, i \in I$ and a topology $\mathcal{T}$ on X, the set

$$\mathcal{T}' = \{U \subset Y : f_i^{-1}(U) \in \mathcal{T}_i \text{ for all } i \in I\}$$

is a topology on Y called the topology **coinduced** by the functions f_i.

2.6 Product topology

In this subsection we explain how to define a topology on the product set of two or more topological spaces. Notably, we concern ourselves also with arbitrary products of a collection of spaces. Hence we could be dealing with an infinite, even uncountable, product of spaces. We recall first what we really mean by a product set.

Note!

Given a collection of sets $X_i, i \in I$ for some index set I, the **product set** $\prod_{i \in I} X_i$ is defined as the set of functions $x : I \to \bigcup_{i \in I} X_i$ such that $x(i) \in X_i$ for all $i \in I$.

The points $x(i)$ are called the **components** of x, and they are usually denoted by x_i for convenience. We also prefer to call x a *point* in the product set rather than draw attention to the fact that it is actually a function. We also shorten the notation of the product set to $\prod_i X_i$ when the index set is clear from context.

In the case of a finite collection of sets $X_i, i = 1, \ldots, n$, this definition is equivalent to the standard Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ given as ordered n-tuples $(x_1, x_2, \ldots, x_n)$.

Note!

If A and B are sets, then A^B denotes the product set $\prod_{b \in B} A$. This is the same thing as the set of functions $f : B \to A$, since every such f corresponds to the element in A^B defined by its components $f_b = f(b)$.

We next endow the product set with a canonical topology called the *product topology.*

Definition 2.14. If $X_i, i \in I$ is a collection of topological spaces, then the **product topology** on $\prod_i X_i$ is defined as the topology induced by the **projetion maps** $\text{proj}_i : \prod_j X_j \to X_i$ defined by $\text{proj}_i(x) = x_i$, i.e. the coarsest topology under which the projection maps are continuous.

In other words, a subbasis for the product topology is given by the sets $\text{proj}_i^{-1}(U)$ for any $i \in I$ and open sets $U \subset X_i$.

We use the term **product space** to refer to a product set with the product topology. The projection maps are sometimes also called **component maps**. We now try to find some intuition on what the product topology actually looks like.

2.6.1 Finite product spaces

Let's start by investigating the product topology in the product of just two topological spaces X and Y.

We try to understand the product topology in $X \times Y$ by figuring out what the basis elements look like. The definition of the induced topology tells us that we should take any open sets $U \subset X$ and $V \subset Y$, and look at the subbasis elements which are of the form

$$\text{proj}_X^{-1}(U) = U \times Y \qquad \text{or} \qquad \text{proj}_Y^{-1}(V) = X \times V.$$

The basis elements of the product topology are then intersections of finitely many of these kinds of sets. But playing around with it a bit, one quickly notices that any finite intersection of these is always a set of the form $U \times V$, with $U \subset X$ and $V \subset Y$ open sets.

The conclusion is that **the basis for the product topology** of $X \times Y$ is simply given as sets of the form $U \times V$, with any choice of open sets $U \subset X$ and $V \subset Y$. The general open sets will then be unions of these kinds of sets.

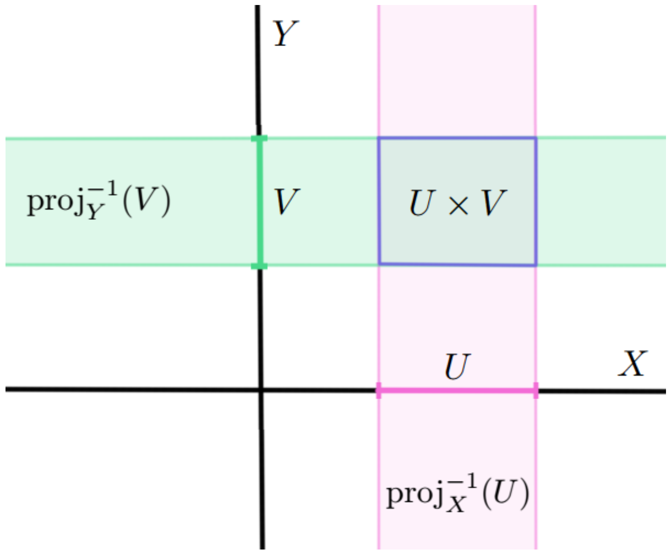


Figure 6: In the product of two spaces, the basis elements of the product topology are given by products of two open sets $U \subset X$ and $V \subset Y$.

In $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, one can apply the same idea to check that the basis for the product topology is given by open rectangles $(a, b) \times (c, d)$. This basis gives the same topology as the basis of open balls, so it doesn't matter whether we define the topology on $\mathbb{R}^2$ as the product topology or the usual metric topology.

For finite products of spaces, i.e. product spaces of the form $X_1 \times X_2 \times \cdots \times X_n$, one can apply this conclusion inductively and obtain that the basis is given by sets of the form $U_1 \times U_2 \times \cdots \times U_n$, with open sets $U_i \subset X_i$.

2.6.2 Infinite product spaces

For product spaces $\prod_{i \in I} X_i$ where the index set I is infinite, the basis elements look a little bit more complicated.

The subbasis elements are still obtained the same way: pick an open set $U \subset X_j$ for some j and take the preimage $\text{proj}_j^{-1}(U) = \{x \in \prod_{i \in I} X_i : x_j \in U\}$ which can be imagined as the infinite product of the set U and the sets X_i for all $i \neq j$.

However, when we find the basis elements we may only take finite intersections of such sets, meaning that in a finite collection of indices i we can restrict ourselves to any open set in X_i, but in the rest of the indices we must keep the whole space X_i. More explicitly, the basis elements can be generated as follows:

- Choose some finite number of indices i_k, $k = 1, \ldots, n$ and open sets $U_{i_k} \subset X_{i_k}$.
- Let $U_j = X_j$ for the rest of the indices j in I with $j \neq i_k$ for any k.
- Define the basis element as $B = \prod_i U_i$.

And any open set in the product topology will then be the union of such basis elements.

It is generally easier to work with the characterization for the product topology given by the component maps. For example, Theorem 2.10 tells us that a sequence x_n in the product space converges to a limit x if and only each component sequence $\text{proj}_i(x_n)$ converges to $\text{proj}_i(x)$.

A characterization for continuous functions into product spaces is obtained from Theorem 2.11. Given a function $f : X \to \prod_i Y_i$ into a product space, we denote by $f_i := \text{proj}_i \circ f$ its **component functions**. The criteria for deciding whether f is continuous can now be written as follows.

Theorem 2.12. *Let $f : X \to \prod_i Y_i$. Then f is continuous if and only if its component functions are continuous for all i.*

The following result sheds some light on the structure of open sets in the product space.

Lemma 2.13. *Let $U \subset \prod_i X_i$ be open. Then $\text{proj}_i U = X_i$ for all $i \in I$ apart from some finite set of indices i.*

Proof. Let $B \subset U$ be some basis element. Then also $\text{proj}_i(B) \subset \text{proj}_i(U)$ for all i. Since $\text{proj}_i(B) = X_i$ for all but finitely many i, we also have $\text{proj}_i(U) = X_i$ for such i. □

Closures also work intuitively in the product topology:

Theorem 2.14. *Let $A_i \subset X_i$ be subsets for all $i \in I$. Then in the product topology on $\prod_{i \in I} X_i$ we have that*

$$\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}.$$

Proof. Let first $x \in \prod_{i \in I} \overline{A_i}$. Pick a basis element $B = \prod_{i \in I} B_i$ in the product topology which contains x (here we know also that $B_i = X_i$ except for finitely many i, but we wont need this fact). Since $x_i \in \overline{A_i}$ for each $i \in I$, there must be a point $y_i \in B_i \cap A_i$ for each $i \in I$ since these intersections are nonempty. But then $y = (y_i)$ is a point in $B \cap \prod_{i \in I} A_i$. Since B was arbitrary, $x \in \overline{\prod_{i \in I} A_i}$.

Let now $x \in \overline{\prod_{i \in I} A_i}$. We show that for each index k and every neighbourhood V_k of x_k, we have that $V_k \cap A_k \neq \emptyset$. To do this, we consider the preimage $U := \text{proj}_k^{-1}(V_k)$, which is a neighbourhood of x and therefore also intersects $\prod_{i \in I} A_i$ at some point y. But now $y_k \in \text{proj}_k(U \cap \prod_{i \in I} A_i) = U_k \cap A_k = V_k \cap A_k$, so we are done since we conclude that $x_k \in \bar{A}_k$. □

We conclude with some examples.

Example 2.7. Let $Q = [0,1]^{\mathbb{N}} := \prod_{i \in \mathbb{N}} [0,1]$ be the **Hilbert cube**. This space happens to be a **homogeneous space**, meaning that for any two points $a, b \in Q$ there is a homeomorphism $f : Q \to Q$ such that $f(a) = b$. Hence topologically every point on the Hilbert cube looks the same up close. This is quite different from finite-dimensional cubes where the topology around a point on the side or corner of the cube looks quite different compared to an interior point. We refer to arxiv:1211.1363 for a proof.

Example 2.8. Let us revisit the topology of pointwise convergence on the space $\mathcal{F}(X, Y)$ of functions $f : X \to Y$. Note that every function in $\mathcal{F}(X, Y)$ can be identified with a point in the product space Y^X. Indeed, every $F \in Y^X$ defines a function $f : X \to Y$ by setting $f(x) = \text{proj}_x(F)$ and vice versa. To be precise, we could define a map $\Phi : \mathcal{F}(X, Y) \to Y^X$ by setting $\Phi(f) = F$.

The topology of pointwise convergence on $\mathcal{F}(X,Y)$ happens to be equivalent with the product topology on Y^X. Indeed, since the product topology is induced by the collection of projections proj_x, a subbasis for the product topology is given by the sets $\text{proj}_x^{-1}(U)$ for each $x \in X$ and open set $U \subset Y$. The set $\text{proj}_x^{-1}(U)$ corresponds exactly to the subbasis element $S(x,U)$ used in the definition of the topology of pointwise convergence at Definition 2.10, making the correspondence given by Φ a homeomorphism between the topology of pointwise convergence on $\mathcal{F}(X,Y)$ and the product topology on Y^X.

2.6.3 Box topology

Another topology on the product space $\prod_i X_i$ is the **box topology** which is defined via the basis of sets $\prod_i B_i$ with $B_i \subset X_i$ open for all i. Compared to the standard product space topology as defined in Definition 2.14, we have omitted the assumption that $B_i \neq X_i$ only for finitely many i. It turns out that omitting the finiteness assumption gives a less useful topology for product spaces, so we do not make much use of the box topology here.

Naturally, the product topology and the box topology are equal if we are considering only finite products (see Section 2.6.1). But for infinite products the box topology gives a finer topology than the standard product topology.

2.7 Exercises

Exercise 2.1. Show that every metric space is a Hausdorff space.

Exercise 2.2. Show that the following are equivalent:

- The space X is T_1, meaning $\forall x, y \in X$, $x \neq y$, there is a neighbourhood V of y such that $x \notin V$.
- Singletons $\{x\}$ are closed sets for every $x \in X$.

Exercise 2.3. Let $\mathbb{R}$ be equipped with the topology given by the basis $\{(-R, R) : R > 0\}$. Show that a sequence has a limit if and only if it is bounded (in the usual sense).

Exercise 2.4. Suppose that $\mathbb{R}$ is equipped with the cofinite topology (A is open if $A = \emptyset$ or A^c is finite). Find all limit points of the sequence $1, 2, 3, \ldots$.

Exercise 2.5. Let X be a topological space in which every open set is also closed. Show that if X is not equipped with the discrete topology, then X cannot be Hausdorff.

Exercise 2.6. Show that in a T_1-space X, a point x is an accumulation point of a set A if and only if every neighbourhood of x contains infinitely many points in A.

Exercise 2.7. Show that any constant function $f : X \to Y$ defined by $f(x) = y$ for all $x \in X$ is continuous.

Exercise 2.8. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Exercise 2.9. Find a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous only at $x = 0$.

Exercise 2.10. Suppose that $\mathcal{B}$ is a basis for Y and $f : X \to Y$ is such that $f^{-1}(B)$ is open for every $B \in Y$. Prove that f is continuous.

Exercise 2.11. Show that if X is a space in which every function $f : X \to \mathbb{R}$ is continuous, then X has the discrete topology.

Exercise 2.12. Let $f, g : X \to Y$ be continuous and Y be Hausdorff.

- Show that the set $\{x \in X : f(x) = y_0\}$ is closed for all $y_0 \in Y$.
- Show that the set $\{x \in X : f(x) = g(x)\}$ is closed.

Exercise 2.13. Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Exercise 2.14. Let $\chi_A : X \to \mathbb{R}$ denote the **characteristic function** of a set $A \subset X$, meaning that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Show that χ_A is discontinuous at a point x if and only if $x \in \partial A$.

Exercise 2.15. Prove the second part of Theorem 2.4: Suppose that X is first-countable. Show that if $A \subset X$ and $a \in \bar{A}$, then there is a sequence in A converging to a.

Exercise 2.16. Show that any continuous function $f : X \to Y$ is sequentially continuous.

Exercise 2.17. Show that $f : X \to Y$ is continuous if and only if for every $x \in X$ and $A \subset X$ such that $x \in \overline{A}$, we have that $f(x) \in \overline{f(A)}$.

Exercise 2.18. Suppose that $f : X \to Y$ is a homeomorphism. Show that if $A \subset X$ then the restriction $f|_A : A \to Y$ is an embedding.

Exercise 2.19. Show that for $A \subset X$ and $B \subset Y$ we have $\overline{A \times B} = \bar{A} \times \bar{B}$ in the product topology of $X \times Y$.

Exercise 2.20. Prove that X is homeomorphic to $X \times \{y\}$ (for any object y).

Exercise 2.21. Show that there exist spaces X, Y such that X can be embedded into Y, Y can be embedded into X, but X and Y are not homeomorphic.

Exercise 2.22. Come up with a continuous open map which is not closed, and vice versa.

Exercise 2.23. Show that the topology of pointwise convergence on $\mathcal{F}(X, Y)$ is just the topology induced by the collection of maps $\chi_x : \mathcal{F}(X, Y) \to Y$ defined by $\chi_x(f) = f(x)$.

Exercise 2.24. Show that if $f : X \to Y$ and X has the topology induced by f, then for any $A \subset X$ we have $\bar{A} = f^{-1}(\overline{f(A)})$.

Exercise 2.25. Let us induce a topology $\mathcal{T}$ on $\mathbb{R}^2$ using the map $f : \mathbb{R}^2 \to \mathbb{R}$ given as $f(x, y) = \sin(x+y)$. Find the closure of $\{(0, 0)\}$ under this topology (draw a picture).

Exercise 2.26. Prove Theorem 2.10.

Exercise 2.27. Prove Theorem 2.11.

Exercise 2.28. Prove that an injective map $f : X \to Y$ is an embedding if and only if f induces the topology on X.

Exercise 2.29. Show that $f : X \to Y$ is closed if and only if the following holds: For every $y \in Y$ and every neighbourhood U of $f^-1(y)$ there exists a neighbourhood V of y such that $f^{-1}(V) \subset U$.

Exercise 2.30. Show that the sequence $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = x^n$ converges pointwise but not uniformly.

Exercise 2.31. Let $f : X \to Y$ be a bijection, and suppose that a topology on Y is given. Induce a topology from Y into X using the function f. Show that $f : X \to Y$ provides a homeomorphism from X with this induced topology into Y.

Exercise 2.32. Let X be a topological space. Show that the topology on X induced by the collection of all continuous maps $f : X \to X$ is just the original topology on X.

Exercise 2.33. Show that the pointwise limit of continuous functions does not need to be continuous (in contrast to the Uniform Limit Theorem).

Exercise 2.34. Let $X = \prod_{i \in I} X_i$. Prove that if $A_i \subset X_i$ for all i and $A := \prod_{i \in I} A_i$, then

$$\bar{A} = \prod_{i \in I} \overline{A_i}.$$

Exercise 2.35. Show that the product of Hausdorff spaces is Hausdorff.

Exercise 2.36. Show that the projection maps $\mathrm{proj}_i : \prod_j X_j \to X_i$ are open maps.

Exercise 2.37. Let the topology on X be induced by a family of maps $f_i : X \to Y_i$. Prove that the map $\Phi : X \to \prod_i Y_i$ defined by $\Phi(x) = \prod_i f_i(x)$ is an embedding.

Exercise 2.38. Verify that the set $\mathcal{T}$ defined in Definition 2.12 is indeed a topology on X. Do the same for the set $\mathcal{T}'$ in Definition 2.13.

Exercise 2.39. Let $f : X \to Y$ be continuous. Show that the graph $G_f = \{(x, f(x)) : x \in X\} \subset X \times Y$ is homeomorphic with X.

Exercise 2.40. Let us say that $f : X \to \mathbb{R}$ is **lower semicontinuous** if the following holds: For every $a \in X$ and every $\epsilon > 0$ there exists a neighbourhood U of a such that $f(x) > f(a) - \epsilon$ for all $x \in U$. Find a topology $\mathcal{T}$ on $\mathbb{R}$ such that $f : X \to \mathbb{R}$ is continuous w.r.t. the topology $\mathcal{T}$ if and only if it is lower semicontinuous.

3 Topological properties

In this section, we highlight certain key properties in topological spaces such as compactness and connectivity.

3.1 Compactness

Note!

Let $\mathcal{D}$ be a collection of subsets of X and $A \subset X$. If $A \subset \bigcup\{D \in \mathcal{D}\}$, we say that $\mathcal{D}$ is a **cover** of A.

If in addition each set in $\mathcal{D}$ is open, we say that $\mathcal{D}$ is an **open cover**.

If $\mathcal{D}' \subset \mathcal{D}$ is also a cover, we say that $\mathcal{D}'$ is a **subcover** of $\mathcal{D}$.

Open covers are the key tool to describe the the concept of *compactness* in topology.

Definition 3.1. A space X is **compact** if every open cover of X has a finite subcover.

In other words, whenever X is covered via some open sets $U_i, i \in I$, we can extract a finite amount $U_{i_1}, \dots, U_{i_N}$ of sets in the cover whose union is still X.

A subset $A \subset X$ of a space is called compact if it is a compact space with respect to the relative topology. This is the same as saying that every open cover of A (via open sets in X) has a finite subcover.

We collect some basic results on compactness in the following theorem.

Theorem 3.1.

- *Every closed subset of a compact space is also compact.*
- *The image of a compact set under a continuous function is compact.*
- *If $A, B \subset X$ are both compact, then so is $A \cup B$.*

Proof. For the first point, let $V' \subset V$ be a closed subset of a compact space. Suppose that $\mathcal{D}$ is an open cover of V'. Then $\mathcal{D}$ might not be an open cover of V, but certainly $\mathcal{D} \cup \{V \setminus V'\}$ is, since $V \setminus V'$ is open due to V' being closed and covers all the parts of V not covered in $\mathcal{D}$. Due to compactness of V there is a finite subcover $\mathcal{D}' \subset \mathcal{D} \cup \{V \setminus V'\}$, and this must then also be a cover of V'. Moreover, the set $\mathcal{D}' \setminus \{V \setminus V'\}$ is a subset of $\mathcal{D}$ and an open finite cover of V'.

For the second point, let $V \subset X$ be a compact set and $f : X \to Y$ be continuous. Let $\mathcal{D}$ be an open cover of $f(V)$. Then $f^{-1}(\mathcal{D}) := \{f^{-1}(U) : U \in \mathcal{D}\}$ is an open cover of V due to continuity. Since V is compact, we obtain a finite subcover $\mathcal{D}' \subset f^{-1}(\mathcal{D})$ which can also be written in the form $\mathcal{D}' = \{f^{-1}(U) : U \in \mathcal{D}''\}$ for some finite subset $\mathcal{D}''$ of $\mathcal{D}$. The set $\mathcal{D}''$ must also be a cover of $f(V)$ since its preimage under f is an open cover of V. Therefore we have found the required finite subcover $\mathcal{D}'' \subset \mathcal{D}$.

Finally, for the last point we may take an open cover of $A \cup B$, use compactness to find two finite subcovers of A and B respectively, and take their union to find a finite subcover of $A \cup B$. This concludes the proof. $\square$

Theorem 3.2. *Let X be a compact space and $f : X \to \mathbb{R}$ be continuous. Then f attains its maximum value in X.*

Proof. The main point is that $f(X)$ is a compact set by Theorem 3.1. Any compact set V in $\mathbb{R}$ contains a maximal element since otherwise the collection $\{(-\infty, x) : x < \sup V\}$ is an open cover without a finite subcover. In the same way we can argue that f attains its minimum on X. $\square$

Since compactness is a property defined using the open sets in X, there is naturally an alternative characterization using closed sets instead. This alternate way to define compactness is sometimes useful, so we present it here as a theorem:

Theorem 3.3. *A topological space X is compact if and only if the following holds: For every collection $\mathcal{F}$ of closed sets in X with $\bigcap_{F\in\mathcal{F}} F = \emptyset$, there exists a finite subcollection $\mathcal{F}' \subset \mathcal{F}$ such that $\bigcap_{F\in\mathcal{F}'} F = \emptyset$.*

In Hausdorff spaces, some additional properties hold for compact sets

Theorem 3.4. *Let X be a Hausdorff space.*

- *If $A, B \subset X$ are compact and disjoint, then A and B have disjoint neighbourhoods.*
- *If $A \subset X$ is compact, it is closed.*
- *The intersection of any family of compact subsets $A_i \subset X$, $i \in I$ is compact.*

Proof. For the first point, we may suppose that neither A or B is empty. For each pair of points $a \in A$, $b \in B$, there exist neighbourhoods $U_b(a)$ of a and $V_a(b)$ of b which are disjoint by the Hausdorff property.

Note that for each $b \in B$, the sets $\{U_b(a) : a \in A\}$ give an open cover of A. Therefore there is a finite subcover $\{U_b(a) : a \in F\}$, where $F \subset A$ is finite. Let

$$\mathcal{U}_b = \bigcup\{U_b(a) : a \in F\} \qquad \text{and} \qquad \mathcal{V}_b = \bigcap\{V_a(b) : a \in F\}.$$

Both of these sets are open, $A \subset \mathcal{U}_b$, $b \in \mathcal{V}_b$, and these two sets are disjoint since $U_b(a)$ and $V_a(b)$ were disjoint.

Now the sets $\mathcal{V}_b$ give an open cover of B, so there is a finite subcover $\{\mathcal{V}_b : b \in F'\}$ for some finite $F' \subset B$. Then the sets

$$U = \bigcap\{\mathcal{U}_b : b \in F'\} \qquad \text{and} \qquad V = \bigcup\{\mathcal{V}_b : b \in F'\}$$

give two open sets which are disjoint and cover A and B respectively.

For the second point, let $x \notin A$. Then $\{x\}$ and A have disjoint neighbourhoods since they are both compact. But this shows that $x \in \text{ext}(A)$ and hence A is closed.

The last part is an exercise. □

3.1.1 Sequential compactness

From metric space theory, one may be used to defining compactness via sequences instead of open covers. We say that a space X is **sequentially compact** if every sequence $(x_n) \in X$ has a convergent subsequence. This is not equivalent with compactness in general topological spaces, and we explore the differences between these two notions further here.

It happens that neither compactness nor sequential compactness implies the other. The examples to prove this are not very simple, however. We provide further details in the appendix at Example A.2 and Example A.3.

The following positive result holds though.

Theorem 3.5. *Let X be compact and $(x_n) \subset X$ be a sequence. Then (x_n) has a cluster point in X.*

Proof. Suppose that no such cluster point exists for a sequence (x_n). Hence every point $a \in X$ has a neighbourhood $U(a)$ such that $x_n \in U(a)$ for only finitely many indices n. Since the sets $U(a), a \in X$ form an open cover of X, there is a finite subcover. But since the sets in this finite subcover only contain finitely many points from the sequence (x_n), the sequence (x_n) should only have a finite amount of points total. This is a contradiction which finishes the proof. □

In first-countable spaces, compactness implies sequential compactness.

Theorem 3.6. *Let X be first-countable and compact. Then X is sequentially compact.*

Proof. Let $(x_n) \subset X$ be a sequence in X. Then by Theorem 3.5, (x_n) has a cluster point. By Theorem 2.4, there is a subsequence converging to that cluster point. □

One should be careful here though, as there are some pathological examples of spaces where the reverse implication doesn't hold: A space can be first-countable and sequentially compact but not compact. See Example A.4 in the appendix.

Finally, we recall the important Heine-Borel theorem which characterizes compactness in Euclidean spaces.

Theorem 3.7 (Heine-Borel theorem)**.** *If $A \subset \mathbb{R}^n$, then A is compact if and only if it is closed and bounded.*

We postpone the proof for now, as this will follow easily from some later results.

3.1.2 Local compactness

Compactness is a topological property which is not enough to be checked locally. For example, the space $\mathbb{R}^n$ is not compact even though any closed ball in $\mathbb{R}^n$ is compact. The concept of *local compactness* turns out to be quite interesting and useful.

Definition 3.2. A space X is **locally compact** if each point $x \in X$ has a neighbourhood U such that $\overline{U}$ is compact.

Naturally, each compact space is locally compact. Local compactness may be characterized in terms of basis elements as well:

Theorem 3.8. *If $\mathcal{B}$ is a basis on a locally compact space X, then for each point $x \in X$ there is a basis element B such that $x \in B$ and $\overline{B}$ is compact.*

Proof. Let $x \in X$. Then x has a neighbourhood U such that $\overline{U}$ is compact. By the definition of a basis there must be a basis element B such that $x \in B \subset U$. But then $\overline{B} \subset \overline{U}$ and since $\overline{B}$ is now a closed set contained in a compact set, it is compact by Theorem 3.1. □

Example 3.1. The space $\mathbb{R}^n$ is locally compact, since for each point $x \in \mathbb{R}^n$ and $r > 0$ the closed ball $\overline{B(x,r)}$ is compact by the Heine-Borel theorem. However, the infinite product $\mathbb{R}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathbb{R}$ is not locally compact.

If $\mathbb{R}^{\mathbb{N}}$ were compact then there would be a basis element, say

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots ,$$

which would be contained within a compact subset. Therefore the closure

$$\overline{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots ,$$

would be compact since it is a closed subset of a compact set. This will quickly lead to a contradiction since there are factors of $\mathbb{R}$ appearing here (in fact, the product of a non-compact space with another non-empty space can never be compact). Take for example

$$U_x = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{n copies}} \times (x, x+1) \times \mathbb{R} \times \cdots \qquad x \in \mathbb{R}.$$

These sets U_x provide an open cover of $\overline{B}$, but there can be no finite subcover as that would mean that a finite amount of intervals $(x, x+1)$ cover $\mathbb{R}$ (the $n+1$:th factor in the product in B).

Subspaces of locally compact spaces need not be locally compact (see Example 3.3), but closed subspaces do inherit local compactness (Exercise 3.7).

We will revisit local compactness again in Section 4.1, where we prove that locally compact Hausdorff spaces satisfy some interesting properties.

3.1.3 Compactification

For various reasons, it is often desirable to take a non-compact space and extend it slightly to turn it into a compact space. This is practically just a process of embedding a space inside a compact space.

Example 3.2. The real line $\mathbb{R}$ is homeomorphic with $S^1 \setminus \{(1,0)\}$, i.e. the unit circle minus a point. This gives an embedding of $\mathbb{R}$ into the compact space S^1. In this embedding the point $(1,0) \in S^1$ can be considered as a 'point at infinity' on $\mathbb{R}$ in the following sense:

Add an 'infinity point' ∞ to $\mathbb{R}$ and create the space $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Extend the embedding from before, call it $F : \mathbb{R} \to S^1$, into a function on $\mathbb{R}^*$ by setting $F(\infty) = (1,0)$. Use this to induce a topology on $\mathbb{R}^*$, which makes $\mathbb{R}^*$ homeomorphic with S^1 since F is a bijection and therefore a homeomorphism in its induced topology.

In more simple terms, the topology on $\mathbb{R}^*$ is simply a topology generated by the usual basis and the sets $\{\infty\} \cup \{x \in \mathbb{R} : |x| > R\}$ for all $R > 0$ (called *neighbourhoods of infinity*). This topology makes $\mathbb{R}^*$ into a compact space.

In this way, adding an extra point to the real line $\mathbb{R}$ can turn it into a compact space using the appropriate topology.

The idea of the previous example can also be used in general Hausdorff spaces.

> **Definition 3.3.** Let X be a Hausdorff space with topology $\mathcal{T}$. Let us pick some object not in X and denote it by ∞, called the point at infinity. Define the set $X^* = X \cup \{\infty\}$. We define a topology $\mathcal{T}^*$ on X^* by setting $\mathcal{T}^* = \mathcal{T} \cup \mathcal{T}_\infty$, where
>
> $$\mathcal{T}_\infty = \{U \subset X^* : \infty \in U \text{ and } X \setminus U \text{ is compact under } \mathcal{T}\}.$$
>
> The space X^* is compact and is called the **one-point compactification** of X.

The verification of X^* being compact in this topology is given as an exercise.

Example 3.3. A subspace of a (locally) compact space does not need to be locally compact in the relative topology. For example, let $\mathbb{Q}^*$ be the one-point compactification of the rational numbers $\mathbb{Q}$ (with the usual subspace topology from $\mathbb{R}$ on $\mathbb{Q}$). Then $\mathbb{Q} \subset \mathbb{Q}^*$ is an open subspace in its own one-point compactification.

However, $\mathbb{Q}$ is not locally compact. In fact, for any irrational number $s \in \mathbb{R} \setminus \mathbb{Q}$, one can choose, say, an increasing sequence of points $(q_n) \subset \mathbb{Q}$ such that $q_n \to s$ as $n \to \infty$. Such a sequence can have no cluster point in $\mathbb{Q}$. However, any basis element $(a,b) \cap \mathbb{Q}$ contains such a sequence since any open interval (a,b) contains an irrational point s.

By Theorem 3.5, any sequence in a compact set has a cluster point. The basis element $(a,b) \cap \mathbb{Q}$ can therefore not be contained in a compact set, so $\mathbb{Q}$ is not locally compact.

In the one-point compactification $\mathbb{Q}^*$, the sequence (q_n) defined above would converge to the point at infinity ∞. Indeed, if U is any neighbourhood of ∞ then U may only omit finitely many of the points in (q_n), as otherwise $\mathbb{Q}^* \setminus U$ could not be compact due to containing infinitely many points in the sequence (q_n) without a cluster point in $\mathbb{Q}$. Hence in the one-point compactification of $\mathbb{Q}$, one can imagine that the point ∞ is used to plug all the holes left by the irrational numbers.

3.2 Connectedness

Definition 3.4. A space X is **disconnected** if there are open sets $U, V \subset X$ (called a **separation** of X) such that

- U and V are nonempty.
- $U \cap V = \emptyset$.
- $X = U \cup V$.

Otherwise we say that X is **connected**.

Hence a set is connected if it cannot be represented as a union of two disjoint non-empty open subsets. A simple alternate characterization is given by the following result.

Lemma 3.9. *A space X is disconnected if and only if there is a continuous surjection $f : X \to \{0, 1\}$.*

As a clarification we note that here the target space $\{0, 1\}$ has the discrete topology.

Note!

It is also common to study whether subsets $A \subset X$ are connected, meaning that A is connected in the relative topology. This can be restated as follows:

A subset $A \subset X$ is connected if there exist no open sets $U, V \subset X$ such that

- $U \cap V \cap A = \emptyset$.
- $U \cap A \neq \emptyset \neq V \cap A$.
- $A \subset U \cup V$.

Intuitively, a subset is connected if it cannot be split into two nonempty parts by disjoint open sets in X.

Connectedness is preserved when taking closures.

Theorem 3.10. *Let $A \subset X$ be connected. Then $\bar{A}$ is connected.*

Another key property of connectedness is that it is preserved when taking images under continuous functions.

Theorem 3.11. *Let $f : X \to Y$ be continuous and X be connected. Then $f(X)$ is connected.*

Proof. If U, V are a separation of $f(X)$, then $f^{-1}(U)$ and $f^{-1}(V)$ are a separation of X since f is continuous. Thus X is disconnected which is a contradiction. Hence $f(X)$ is connected. ☐

Example 3.4. Let $f : I \to \mathbb{R}$ be a continuous function on some interval $I \subset \mathbb{R}$. Then the graph $G_f = \{(x, f(x)) : x \in I\}$ is a connected set due to the fact that it is the image of the connected set I under the continuous map $x \mapsto (x, f(x))$.

A key consequence of Theorem 3.11 is the Intermediate Value Theorem:

Theorem 3.12 (Intermediate Value Theorem)**.** *Let X be connected and $f : X \to \mathbb{R}$ be continuous. If $a, b \in X$, then f takes every value between $f(a)$ and $f(b)$ in X.*

The next result shows that the following intuitive fact is true: if two connected sets have a common part, then joining them together still produces a connected set.

Theorem 3.13. *Let $A_i, i \in I$ be any collection of connected subsets of X, and $\bigcap_{i\in I} A_i \neq \emptyset$. Then $\bigcup_{i\in I} A_i$ is connected.*

Multiplying connected spaces together still gives a connected space, even if the product is infinite.

Theorem 3.14. *Let $X_i, i \in I$ be a collection of connected spaces. Then the product space $\prod_i X_i$ is connected in the product topology.*

Proof. We proceed with a proof by contradiction and assume that there exists a separation of $\prod_i X_i$ with open sets U, V. By Lemma 2.13, there is some finite set of indices $K \subset I$ such that $\mathrm{proj}_i(U) = X_i$ and $\mathrm{proj}_i(V) = X_i$ for $i \notin K$. Hence we can project down to only the indices in K and think of the sets U, V as a separation of the finite product space $\prod_{i\in K} X_i$, so it is enough to prove the connectedness of finite products of connected spaces here.

Let's write $K = \{1, \ldots, N\}$ for some $N \in \mathbb{Z}_+$. Let us consider a point $a = (a_i)_{i=1}^N \in \prod_{i=1}^N X_i$. For each index j we can define the set

$$A_j := \{a_1\} \times \cdots \times \{a_{j-1}\} \times X_j \times \{a_{j+1}\} \times \cdots \times \{a_N\}.$$

This is a subset of the product space which is connected, and homeomorphic to X_j (essentially Exercise 2.20). Hence A_j is connected, and each of the A_j's contains the point a. Therefore by Theorem 3.13, the union $A = \bigcup_{j=1}^n A_j$ is connected. One can think of the set A as the "coordinate axes" of the product space, with the point a being the origin.

Now we would like A to intersect both of the sets U and V to arrive at a contradiction, since then $U \cap A$ and $V \cap A$ would separate the connected set A, but this requires choosing the point a appropriately. To find the right point a, we need to find two points $u \in U$ and $v \in V$ such that $u_i = v_i$ except for one index $i = j$ (and choose either of them as the point a). Pick first any two points $u^* \in U$ and $v^* \in V$. If u^* and v^* differ in more than one component, then let k be such a component and define a new point w by setting $w_i = u_i$ for $i \neq k$ and $w_k = v_k$. Then w differs from u^* in only one component, and from v^* in one less component than u^* did. If $w \in V$, then u^* and w are the points we were looking for. Otherwise we can continue this process with w and v^* and find the right points in finitely many steps. □

3.2.1 Connected components

Any space may be broken down to its maximal connected subsets, called connected components.

Definition 3.5. A subset $A \subset X$ is called a **connected component** or simply **component** of the space X if A is connected and every set B satisfying $A \subsetneq B \subset X$ is disconnected.

The following result details the basic properties of connected components.

Theorem 3.15.

(1) Connected components are closed in X.

(2) If $A, B \subset X$ are connected components, then either $A = B$ or $A \cap B = \emptyset$.

(3) We may write X as a disjoint union

$$X = \bigcup \{A : A \text{ is a connected component of } X\}.$$

(4) Given $x \in X$, the component x belongs to is given by

$$C_x = \bigcup \{A : x \in A \subset X,\ A \text{ is connected}\}.$$

In other words, the connected components of X provide a decomposition of the space X where each point lies in exactly one component.

Example 3.5. The set of rational numbers $\mathbb{Q}$ has infinitely many components, each being a singleton $\{q\}$. This is due to the fact that whenever $x \in \mathbb{Q}^c$ is irrational the sets $(-\infty, x) \cap \mathbb{Q}$ and $(x, \infty) \cap \mathbb{Q}$ give a disjoint open cover of $\mathbb{Q}$. By choosing x appropriately one can use this to show that any subset of $\mathbb{Q}$ containing more than one point is disconnected.

Note, however, that singletons are not open sets in $\mathbb{Q}$ since the intersection of any open interval with $\mathbb{Q}$ must contain infinitely many points in $\mathbb{Q}$. Hence the components of $\mathbb{Q}$ are not open sets.

As a counterpart to this example, the following theorem holds.

Theorem 3.16. *If a space has a finite amount of components, they are open sets.*

Proof. Components are closed, and finite unions of closed sets are closed. Since each component is the complement of the union of the other components, the result follows. ☐

3.2.2 Path connectedness

Let $I = [0, 1]$. A continuous map $\gamma : I \to X$ is called a **path**. Note that the image $\gamma(I)$ is always connected due to Theorem 3.11.

If $x, y \in X$ and γ is a path such that $\gamma(0) = x$ and $\gamma(1) = y$, then we say that γ **connects** x and y.

Definition 3.6. A space X is **path connected** if for every pair of points x, y, there is a path $\gamma : I \to X$ which connects x and y.

A space which is path connected is always connected (exercise). However, the converse is not true.

Example 3.6. Let X be the *topologist's sine curve*:

$$X = \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0, 1] \right\} \cup \{(0, 0)\}.$$

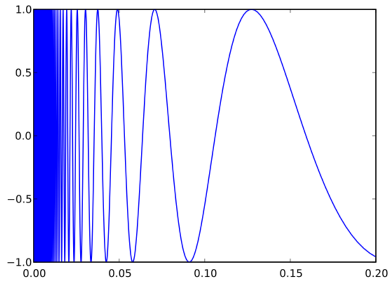


Figure 7: The topologist's sine curve

This space is not path connected, since due to the discontinuity of the function $f(x) = \sin \frac{1}{x}$ at $x = 0$ there is no way to connect the point $(0, 0)$ to any other point on X via a continuous path. However, there is no separation of X because any neighbourhood of $(0, 0)$ must also contain other parts of X, and hence any separation of X must also be a separation of the graph of $\sin \frac{1}{x}$ on $(0, 1]$ which is connected due to being the graph of a continuous function on the connected set $(0, 1]$.

3.2.3 Local connectedness

Definition 3.7. A space X is **locally connected** if for every point $x \in X$ and every neighbourhood U of x, there exists a connected neighbourhood V of x such that $V \subset U$.

Example 3.7. A space may be connected but not locally connected. The *topologist's comb* is defined as the planar set

$$X = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \cup \bigcup_{n \in \mathbb{Z}_+} \left\{ \frac{1}{n} \right\} \times [0,1].$$

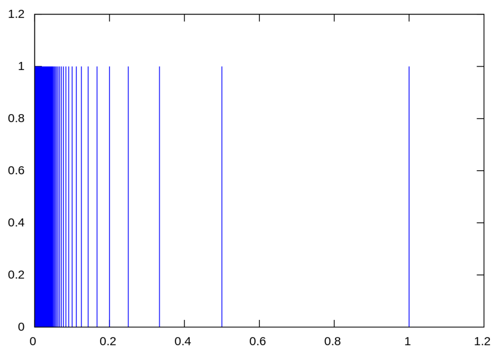


Figure 8: The topologist's comb

This space is even path connected due to the fact that any points on the 'teeth' of the comb may be joined together by travelling down and through the horizontal line segment $[0,1] \times \{0\}$. However, it fails to be locally connected as for each $t > 0$, any small neighbourhood of the point $(0,t) \in X$ will consist of infinitely many connected components.

3.3 Exercises

Exercise 3.1. Let $a \in X$ and suppose that $X \setminus U$ is compact whenever U is a neighbourhood of a. Prove that X is compact.

Exercise 3.2. Show that any finite set is compact, but a countable set doesn't need to be compact.

Exercise 3.3. Prove Theorem 3.3.

Exercise 3.4. Let X be compact, Y be Hausdorff, and $f : X \to Y$ be continuous.

- Prove that f is closed (the image of a closed set is closed under f).
- Prove that if f is a bijection then it is a homeomorphism.

Exercise 3.5. Suppose that X is equipped with the cofinite topology ($U \subset X$ is open if $U = \emptyset$ or U^c is finite). Prove that every subspace $A \subset X$ is compact.

Exercise 3.6. Suppose that $\mathcal{B}$ is a basis for the topology on X. Suppose that whenever $\mathcal{D} \subset \mathcal{B}$ is a cover of X, it has a finite subcover. Prove that X is compact.

Exercise 3.7. Show that a closed subspace of a locally compact space is locally compact.

Exercise 3.8. Prove Theorem 3.10.

Exercise 3.9. Prove Lemma 3.9.

Exercise 3.10. Let A be connected. Is $\operatorname{int}(A)$ necessarily connected?

Exercise 3.11. Prove part (1) of Theorem 3.15.

Exercise 3.12. Let $x \in X$. Show that the connected component C_x of x as defined in Theorem 3.15 is indeed a connected set, and that it is not contained in any larger connected subset of X.

Exercise 3.13. Suppose that X is **totally disconnected**, meaning that its connected components are singletons. Is the topology on X necessarily the discrete topology?

Exercise 3.14. Find a topology on $\{0, 1\}$ which is connected but not path-connected.

Exercise 3.15. Show that the product of totally disconnected spaces is totally disconnected.

Exercise 3.16. Let X and Y be connected. Prove a special case of Theorem 3.14 by showing that $X \times Y$ is connected.

Exercise 3.17. Verify that the space X^* given in Definition 3.3 is compact.

Exercise 3.18. Prove the last part of Theorem 3.4: In a Hausdorff space X, the intersection of any family of compact subsets $A_i \subset X$, $i \in I$ is compact.

Exercise 3.19. Let X be locally compact and $f : X \to Y$ continuous. Is $f(X)$ necessarily locally compact? What if f is also an open mapping?

Exercise 3.20. Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\} \cup \{\frac{1}{n} : n \geq 1\}$ of $\mathbb{R}$.

Exercise 3.21. Show that if a metric space X is connected and has more than one point, it is uncountable.

Exercise 3.22. Comparing to the previous exercise, find a space X in which the intersection of two compact sets $A, B \subset X$ is not compact.

Exercise 3.23. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function such that $|f(x)| \to \infty$ as $|x| \to \infty$. Prove that f is a closed map.

Exercise 3.24. Prove that S^1 and S^2 are not homeomorphic.

Exercise 3.25. Find two maps $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ which induce the standard topology on $\mathbb{R}^2$ from the standard topology on $\mathbb{R}$. Is is possible to do this with only one map?

Exercise 3.26. Let $X \neq \emptyset$ be compact and Hausdorff. Given a continuous function $f : X \to X$, show that there exists a closed subset $A \subset X$ such that $f(A) = A$.

4 Key results

In this section, we aim to take on some more challenging topological results which have broader impact later on. One starting motivation here could be to answer the following question:

Under which conditions is the topology on a space X given by some metric on X?

Topological spaces whose topology is given by a metric are called **metrizable spaces**. Of course, these are the same thing as metric spaces, but the difference in terminology here is to highlight the fact that we want to start with some space in which we do not a priori have a metric, and somehow discover a metric which generates the topology on that space through its other topological properties.

In finding the answer to questions like this, we also come across some new important concepts and key results. The first main properties of metrizable spaces we would like to highlight are their *separation* and *countability* properties. For example, in previous spaces we noted that metric spaces are Hausdorff and first-countable. Let us first take a deeper look at various related properties for topological spaces.

4.1 The separation axioms

In subsection Section 2.1 we introduced separation conditions such as T_1*-spaces* and T_2*/Hausdorff spaces*. We give some further classification on related concepts:

Definition 4.1 (Separation Axioms)**.** Let X be a topological space. Then:

- X is T_0 or **Kolmogorov** if for each $x, y \in X$, $x \neq y$, there is either a neighbourhood of x not containing y, or a neighbourhood of y not containing x. That is to say, x and y have distinct sets of neighbourhoods.
- X is T_1 or **Fréchet** if for each $x, y \in X$, $x \neq y$, there is a neighbourhood U of x such that $y \notin U$.
- X is T_2 or **Hausdorff** if for each $x, y \in X$, $x \neq y$, there exist neighbourhoods U of x and V of y such that $U \cap V = \emptyset$.
- X is **regular** if for each $x \in X$ and each closed set $F \subset X$ with $x \notin F$, there exist neighbourhoods U of x and V of F such that $U \cap V = \emptyset$.
- X is T_3 if it is regular and Hausdorff.
- X is **normal** if for each pair of closed sets $F, G \subset X$ with $F \cap G = \emptyset$, there exist neighbourhoods U of F and V of G such that $U \cap V = \emptyset$.
- X is T_4 if it is normal and Hausdorff.

Naturally, these properties are in order of superiority, e.g. T_0 is the weakest condition and T_4 is the strongest. The main new concepts here are **regular** and **normal** spaces. Both of these have an important alternative formulation:

Lemma 4.1. *Let X be a topological space.*

- *X is regular if and only if for every open set U and point $x \in U$, there is a neighbourhood V of x such that $\bar{V} \subset U$.*
- *X is normal if and only if for every open set U and closed set $A \subset U$, there is a neighbourhood V of A such that $\bar{V} \subset U$.*

Proof. We prove the first item, the proof for the second item is similar. Let first X be regular and $x \in U$ with U open. then, picking disjoint neighbourhoods V and W of x and U^{c}, we claim that $\bar{V} \subset U$. This is true because $U^{\mathsf{c}} \subset W \subset V^{\mathsf{c}}$.

On the other hand, if the condition is true then for any point x and closed set F not containing x, we can find a neighbourhood V of x such that $\bar{V} \subset F^{\mathsf{c}}$. Thus V and $\bar{V}^{\mathsf{c}}$ are the separate neighbourhoods of x and F we are looking for. $\square$

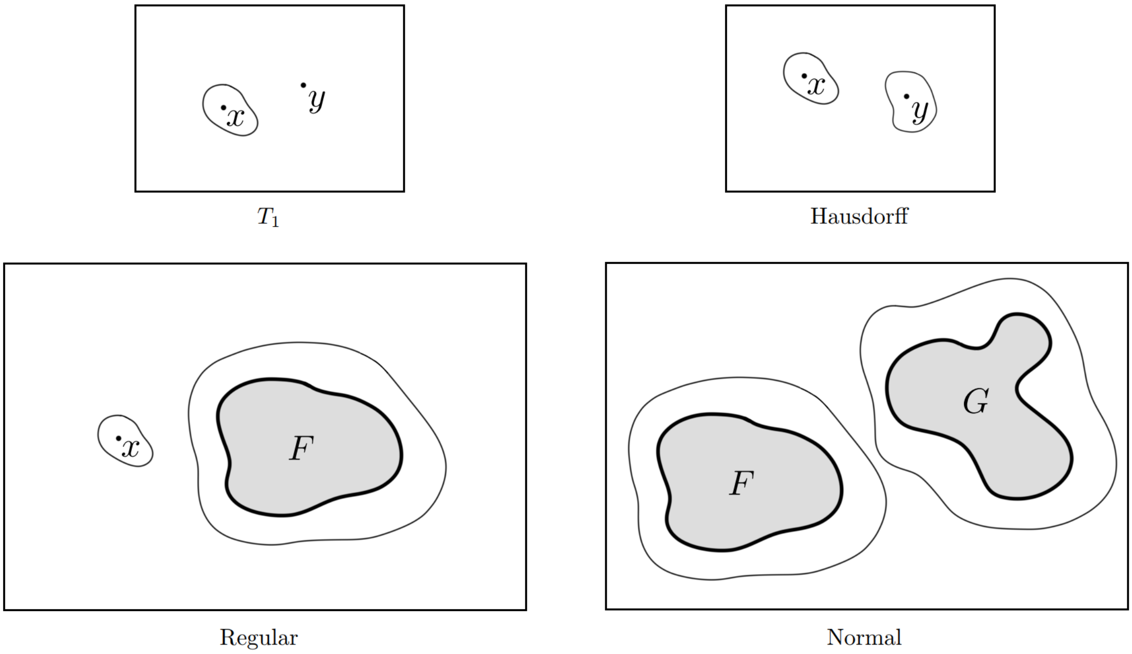


Figure 9: The separation axioms visualized.

The condition of a space being normal is quite strong, and we first discover that all metric spaces are normal:

> **Theorem 4.2.** *Let X be a metrizable space. Then X is normal.*

Proof. Let $F, G \subset X$ be closed and disjoint. For each $x \in F$, since x is in the exterior of G we can choose a radius $r_x > 0$ so that the ball $B(x, r_x)$ does not intersect G. Similarly, for all $y \in G$ we pick $r_y > 0$ so that $B(y, r_y) \cap F = \emptyset$. Then the sets

$$U = \bigcup_{x \in F} B(x, r_x/2) \quad \text{and} \quad V = \bigcup_{y \in G} B(y, r_y/2)$$

give neighbourhoods of F and G. These neighbourhoods are disjoint due to the triangle inequality: If $z \in B(x, r_x/2) \cap B(y, r_y/2)$ for some $x \in F$ and $y \in G$, then

$$d(x, y) \leq d(x, z) + d(z, y) \leq \frac{r_x}{2} + \frac{r_y}{2} \leq \max\{r_x, r_y\}.$$

This would imply that $x \in B(y, r_y)$ or $y \in B(x, r_x)$ depending on which radius of r_x and r_y was bigger, which is a contradiction. Hence U and V give the desired disjoint neighbourhoods. □

Next, we go through some basic properties of Hausdorff and regular spaces.

> **Theorem 4.3.**
> - *Subspaces of a Hausdorff space are Hausdorff. Products of Hausdorff spaces are Hausdorff.*
> - *Subspaces of a regular space are regular. Products of regular spaces are regular.*

Proof. The first item is an exercise. We prove the second item about regular spaces.

Let first X be regular and $A \subset X$. Let's take a closed set $F \subset A$ (in the relative topology on A) and $x \in A \setminus F$. We note that $A \setminus F = U \cap A$ for some $U \subset X$ which is open in the original topology of X, and by Lemma 4.1 there exists a neighbourhood V of x such that $\bar{V} \subset U$.

Then $V \cap A$ and $\bar{V}^c \cap A$ are disjoint neighbourhoods of x and F in the relative topology on A, so A is regular in the relative topology.

Let then $X_i, i \in I$ be some regular spaces, and let $X = \prod_{i \in I} X_i$ be their product space. Let $x \in X$ be a point and $U \subset X$ be a neighbourhood of x. Pick a basis element $B = \prod_{i \in I} B_i$ such that $x \in B \subset U$, and hence there is some finite index set $J \subset I$ such that $B_i = X_i$ for $i \notin J$.

For each $j \in J$, we use the regularity of X_j to pick a neighbourhood V_j of x_j such that $\bar{V}_j \subset B_j$. Then if we define $V_i = X_i$ for $i \notin J$, we find that $V = \prod_{i \in I} V_i$ is a neighbourhood of x such that $\bar{V} \subset U$, since $\bar{V} = \overline{\prod_{i \in I} V_i} = \prod_{i \in I} \bar{V}_i$ by Theorem 2.14 and $\prod_{i \in I} \bar{V}_i \subset \prod_{i \in I} B_i \subset U$. □

However, the product of normal spaces does not need to be normal, as shown by the following example.

Example 4.1. Let $\mathbb{R}_\ell$ denote the real line equipped with the topology generated by the basis of half-open intervals $[a, b)$ with $a, b \in \mathbb{R}$ and $a < b$. This is also called the **lower limit topology** on $\mathbb{R}$, and it is a finer topology than the standard topology on $\mathbb{R}$.

The space $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$ is called the **Sorgenfrey plane**. This space is not a normal space even if $\mathbb{R}_\ell$ is normal. We provide the full details of the argument in the appendix at Case A.1, but here is the summary:

A basis for the product topology on $\mathbb{R}_\ell^2$ is given by rectangles of the form $[a, b) \times [c, d)$. Let $\Delta = \{(x, -x) : x \in \mathbb{R}\}$ denote the anti-diagonal in $\mathbb{R}_\ell^2$. Any subset of Δ happens to be closed. In particular the sets $F = \{(x, -x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ and $\Delta \setminus F$ are both closed sets. However, it can be shown that there are no two neighbourhoods of F and $\Delta \setminus F$ which are disjoint. Thus $\mathbb{R}_\ell^2$ is not normal.

We go through some more examples.

Example 4.2. The space $\{0, 1\}$ with topology $\{\emptyset, \{0\}, \{0, 1\}\}$ is T_0 but not T_1.

Example 4.3. Any uncountable space with the cofinite topology (a set U is open if and only if U is empty or U^c is finite) is an example of a topological space which is T_1 but not Hausdorff.

Example 4.4. We find a space which is Hausdorff but not regular. Let $\mathbb{R}_K$ denote the real line equipped with a topology with a basis consisting of both the open intervals (a, b) and sets of the form $(a, b) \setminus K$ with $K = \{1/n : n \in \mathbb{Z}_+\}$. This space is Hausdorff since the topology is finer than the usual topology on $\mathbb{R}$, but it is not regular since the set K is closed but there is no two neighbourhoods of 0 and K which don't intersect. Indeed, any neighbourhood of 0 must contain at least the set $(-\epsilon, \epsilon) \setminus K$ for some small $\epsilon > 0$, and therefore must intersect any neighbourhood of the point $1/n \in K$ whenever $1/n < \epsilon$.

Example 4.5. The Sorgenfrey plane from Example 4.1 is regular but not normal. It is regular because it is the product of two regular spaces (exercise), but we showed that it is not normal.

We conclude this subsection with an important result showing that compactness can enhance the separation properties of a space.

Theorem 4.4.
- *Compact Hausdorff spaces are normal.*
- *Locally compact Hausdorff spaces are regular.*

The proofs of these two facts are exercises, though we remark that the first point is a consequence of Theorem 3.4.

4.2 The countability axioms, dense sets, and more

In Definition 2.4, we already encountered the first *axiom of countability* by defining spaces to be first-countable if every point has a countable neighbourhood basis. In this subsection we highlight some further properties related to countability, starting from the second axiom of countability:

Definition 4.2. A topological space X is **second-countable** if it has a countable basis.

Second-countability is a much stricter requirement than first-countability. Metric spaces are always first-countable, but need not be second-countable. Euclidean spaces $\mathbb{R}^n$ are second-countable since one can use balls $B(q, r)$ with rational centers $q \in \mathbb{Q}^n$ and radii $r \in \mathbb{Q}$ as a countable basis.

Example 4.6. Let $\mathbb{R}_\ell$ denote the real line equipped with the topology generated by the basis of half-open intervals $[a, b)$ with $a, b \in \mathbb{R}$ and $a < b$. Then $\mathbb{R}_\ell$ is first-countable due to every point x having a countable neighbourhood basis given by $[x, x + 1/n), n \in \mathbb{Z}_+$. However, it is not second-countable: For each $x \in \mathbb{R}$, there must be some basis element contained in $[x, x + 1)$ which contains the point x, call this B_x. If $x < y$, then $B_x \neq B_y$ because $B_y \subset [y, y + 1)$ and therefore B_y cannot contain the point x. Thus the collection $\{B_x : x \in \mathbb{R}\}$ contains a different set for each x in the uncountable set $\mathbb{R}$, so it is uncountable.

In second-countable spaces, the two forms of compactness we encountered in Section 3.1.1 become equivalent.

Theorem 4.5. *Let X be second-countable. Then X is compact if and only if it is sequentially compact.*

The proof is an exercise, but using this we can easily prove the Heine-Borel theorem, Theorem 3.7:

Proof of Theorem 3.7: Let $A \subset \mathbb{R}^n$ be closed and bounded. We first claim that A is sequentially compact, so take a sequence $(x_k) \subset A$. Since (x_k) is bounded, we can extract a subsequence converging in $\mathbb{R}^n$ as follows: The sequence (x_k) is bounded in each component, and any bounded sequence in $\mathbb{R}$ has an increasing or decreasing subsequence. Since increasing/decreasing subsequences which are bounded in $\mathbb{R}$ have a limit, also (x_k) has a limit. Since A is closed, such a limit is necessarily in A.

Now since $\mathbb{R}^n$ is second-countable, so are its subsets and hence Theorem 4.5 implies that A is compact since it is sequentially compact.

The "only if" part of the theorem is also simple to prove: If A is not bounded we can easily extract an unbounded sequence without a converging subsequence, and if A is not closed we can extract a sequence converging to a point not in A, so in both cases A cannot be compact as it is not sequentially compact. $\square$

Definition 4.3. A space X is **Lindelöf** if every open cover of X has a countable subcover.

A second-countable space is Lindelöf, though the converse may not hold (the lower limit topology shown in the previous example shows this, but takes a bit of work).

Second-countability can be used in many different ways, we make note of one key result which will be useful later in the proof of the Urysohn metrization theorem.

Theorem 4.6. *A second-countable regular space is normal.*

Proof. Let X be the space, and $A, B \subset X$ be closed. For each $x \in A$, we use regularity to argue that there exists a basis element $U(x)$ such that $x \in U(x)$ and $\overline{U(x)} \cap B = \emptyset$. Since the basis can be assumed to be countable, we obtain an open cover of A via a countable number of basis elements $U_1, U_2, \ldots$ such that $\overline{U_n} \cap B = \emptyset$ for all n.

Similarly, we can get an open cover $V_1, V_2, \ldots$ of B such that $\overline{V_n} \cap A = \emptyset$ for all n. It would be nice if the sets $U = \bigcup_n U_n$ and $V = \bigcup_n V_n$ were disjoint neighbourhoods of A and B, but this is not always the case. Instead, we perform a clever trick by considering the sequences

$$U_n' = U_n \setminus \bigcup_{k=1}^{n} \overline{V_n} \quad \text{and} \quad V_n' = V_n \setminus \bigcup_{k=1}^{n} \overline{U_n}.$$

These two sequences of sets also give open covers of A and B respectively, and now $U' = \bigcup_n U'_n$ and $V' = \bigcup_n V'_n$ will be disjoint. The disjointness follows from the fact that if $x \in U' \cap V'$, then $x \in U'_n$ and $x \in V'_m$ for some n, m. Let's say $n \geq m$ (the case $n \leq m$ is similar), in which case by the definition of U'_n we have that $x \in U_n$ but $x \notin V_m$, which is a contradiction to the fact that $x \in V'_m$. Hence U' and V' are disjoint neighbourhoods of A and B, so X is normal. □

4.2.1 Density and separability

In many practical situations, we are interested in approximating some mathematical object by a collection of more familiar objects within a given space. To make this idea more precise we introduce the notion of dense subsets of a topological space.

> **Definition 4.4.** A set $S \subset X$ is **dense** in X if every nonempty open set contains a point in S.

In other words, S is dense if and only if $\bar{S} = X$ (exercise). Intuitively, a dense set provides a collection of elements of X which can be chosen arbitrarily close to any other element of x.

Example 4.7. The rational numbers $\mathbb{Q}$ constitute a dense subset of the real line $\mathbb{R}$, since every open interval contains a rational number.

Example 4.8. The set of smooth functions $f : (0,1) \to \mathbb{R}$, denoted by $C^\infty((0,1))$, is a dense subset of the set of continuous functions on $(0,1)$, denoted $C((0,1))$, equipped with the topology of uniform convergence. We omit the proof here as this is more of an exercise in calculus/real analysis.

Spaces where a countable dense subset can be found are of special interest.

> **Definition 4.5.** A space X is **separable** if there is a countable dense subset.

Any second-countable space is separable, since a countable dense subset may be found simply by picking one point x_n inside each basis element B_n, $n \in \mathbb{N}$.

In this section, we showed that second-countability implies each of the following three: first-countability, the Lindelöf condition, and separability. In general, neither of these three imply the other, but in metric spaces the situation is a bit different:

> **Theorem 4.7.** *Let X be a metrizable space. Then the following are equivalent:*
> - *X is second-countable.*
> - *X is Lindelöf.*
> - *X is separable.*
> - *For each $\epsilon > 0$, X has a countable open cover consisting of sets of diameter $< \epsilon$.*

4.3 Urysohn's lemma

We next start proving the key results in this section. The first result is Urysohn's lemma, which is an incredibly useful tool for proving further topological results and particularly gives some concreteness to the structure of normal spaces.

> **Theorem 4.8** (Urysohn's lemma)**.** *Let X be a normal space, and let $A, B \subset X$ be disjoint and closed. Then there exists a continuous function*
>
> $$f : X \to [0,1]$$
>
> *such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

The fact that X here is a normal space implies that the disjoint closed sets A, B have disjoint neighbourhoods. Urysohn's lemma gives a very concrete way to realize this separation via a continuous function that maps these sets into separate points on the real line. From such a map f one can always recover disjoint neighbourhoods of A and B by taking, say, the preimages of $[0, 1/3)$ and $(2/3, 1]$ under f. Finding such an f from scratch is no simple task, but there is a clever proof involving inductive iteration of the normality property which we present here.

Proof. We may suppose that neither A nor B are empty. Let $I = [0, 1]$.

Let $Q \subset I$ be the set of rational numbers in I.

Let us first use the property that X is normal to pick a neighbourhood $U(0)$ of A such that $\overline{U(0)} \cap B = \emptyset$. Let $U(1) = X \setminus B$ so that $U(1)$ is also a neighbourhood of A and $\overline{U(0)} \subset U(1)$.

We will now construct, for each $q \in Q$, a neighbourhood $U(q)$ of A and we do this in a way where $\overline{U(q)} \subset U(r)$ whenever $q < r$. The proof of this will be done by induction as follows:

Existence of neighbourhoods $U(q)$: We use the fact that Q is countable to order this set as a sequence:

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots$$

Here the ordering is based first on denominator, then on numerator. We now apply induction to construct the sets $U(q)$. The initial step is done since $U(0)$ and $U(1)$ are already chosen.

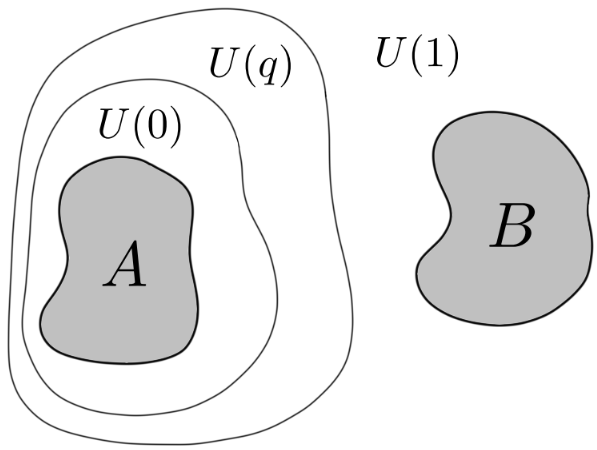


Figure 10: Picking a nested sequence of neighbourhoods $U(q)$ of A.

Let now $q = \frac{r}{s}$. We may suppose that $U(q)$ is chosen for all rational numbers $\frac{r'}{s'}$ with $s' < s$ and those of the form $\frac{r'}{s}$ with $r' < r$. We then need to choose $U(q)$.

Pick $q_- \in Q$ as the biggest number for which $U(q_-)$ was already chosen and $q_- < q$. Do the same to pick a smallest q_+ for which $q < q_+$ and $U(q_+)$ has already been chosen.

By induction assumption, $\overline{U(q_-)} \subset U(q_+)$ and hence we may use normality to choose a neighbourhood $U(q)$ of $\overline{U(q_-)}$ which does not intersect the set $X \setminus U(q_+)$. Hence $\overline{U(q_-)} \subset U(q)$ and $\overline{U(q)} \subset U(q_+)$. This finishes the induction argument and therefore gives us the appropriate "increasing" family of open sets $U(q)$ in the parameter q.

Definition and continuity of f: We may now define our function f by setting $f(x) = 1$ for $x \in B$ and

$$f(x) = \inf\{q \in Q : x \in U(q)\} \text{ for } x \in X \setminus B.$$

Then $f : X \to [0, 1]$ indeed maps all points in A to 0 and all points in B to 1.

We should verify that f is continuous. We claim first that

$$f^{-1}([0, t)) = \bigcup\{U(q) : q \in Q, q < t\} \qquad \text{for all } t \leq 1.$$

"$\supset$": If x is in the union then $x \in U(q)$ for some $q < t$, and $f(x) = \inf\{q' \in Q : x \in U(q')\} \leq q < t$.

"$\subset$": If $t > f(x) = \inf\{U(q) : q \in Q, q < t\}$, then there exists some $q < t$ such that $x \in U(q)$, so x is in the union.

Thus at least the preimages of the intervals $[0, t)$ are open. We next claim that

$$f^{-1}([0,t]) = \bigcap\{\overline{U(q)} : q \in Q, q > t\} \qquad \text{for all } t < 1.$$

"$\supset$": Let $x \in \overline{U(q)}$ for all $q > t$. Now for any $q' > q$ we have $\overline{U(q)} \subset U(q')$ and therefore $x \in U(q')$. Since q' can be chosen arbitrarily close to t, we have $f(x) \le t$ immediately by definition of f.

"$\subset$": If $f(x) \le t$, then for every $\epsilon > 0$ we must have some q with $q < t + \epsilon$ with $x \in U(q)$, hence also $x \in \overline{U(q)}$. Since the sequence of $\overline{U(q)}$:s is increasing in q, we also have $x \in \overline{U(q')}$ for all $q' \ge q$. Since ϵ was arbitrary, $x \in \overline{U(q)}$ for all $q > t$.

Thus the preimages of the closed intervals $[0, t]$ are closed. Now we simply find that

$$f^{-1}((t_1,t_2)) = f^{-1}([0,t_2)) \setminus f^{-1}([0,t_1]) \qquad \text{for all } t_1 < t_2 < 1,$$

which is an open set since it is an open set minus a closed set. Since also $f^{-1}([0,1]) = X$, we have that the preimage of any open set under f is open. Thus f is continuous. □

4.4 The Urysohn metrization theorem

In this section we prove an important result regarding metrization. Before we get to it, we must provide one important example of a metrizable space.

Example 4.9. The Hilbert cube $Q = [0,1]^{\mathbb{N}}$ is metrizable. As a metric, we may use the map $d : Q \times Q \to \mathbb{R}$ defined by

$$d(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n}|x_n - y_n|.$$

It is simple to verify that this satisfies the axioms of a metric. Let us check that the topology given by d is the same as the product topology on Q. To check this, we use the characterization of the product topology as the topology induced by the projection maps $\mathrm{proj}_n : Q \to [0,1]$ in combination with Theorem 2.11.

First of all, the projection maps proj_n are clearly continuous in the metric d, since if $d(x,y) \le \epsilon$, then $|x_n - y_n| \le 2^n\epsilon$. On the other hand, if $f : X \to (Q,d)$ is a map from some topological space X to Q with the metric topology, and we assume that $\mathrm{proj}_n \circ f : X \to [0,1]$ is continuous for all $n \in \mathbb{N}$, then f is continuous into the metric topology due to the following reasoning.

Given $x \in X$ and $\epsilon > 0$, we show that there is a neighbourhood U of x such that $d(f(y), f(x)) < \epsilon$ for all $y \in U$. Since $\mathrm{proj}_n \circ f$ is continuous for each n, there is for each n a neighbourhood U_n of x such that $|f(y)_n - f(x)_n| < \epsilon/2$ whenever $y \in U_n$. Pick now N large enough so that $\sum_{n=N+1}^{\infty} 2^{-n} < \epsilon/2$. Then let $U = U_0 \cap U_1 \cap \cdots \cap U_N$. This is a neighbourhood of x, and if $y \in U$ then

$$d(f(x), f(y)) = \sum_{n=0}^{\infty} \frac{1}{2^n}|f(x)_n - f(y)_n| < \frac{\epsilon}{2}\sum_{n=0}^{N} 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves the claim.

Using the same idea as in the previous example, one can easily prove the following result.

Theorem 4.9. *A countable product of metric spaces is metrizable.*

We are now ready to tackle the Urysohn metrization theorem.

Theorem 4.10 (Urysohn metrization theorem)**.** *Every second-countable T_3-space is metrizable.*

Proof. Let X be T_3 (Hausdorff + regular) and $\{B_n\}$ be a countable basis on X. We should first note that the space X is actually also a normal space because of Theorem 4.6, so we are in a position to apply Urysohn's lemma.

We will show that X is metrizable by embedding X inside the Hilbert cube $Q = [0,1]^{\mathbb{N}}$, which is a metrizable space by Example 4.9. The idea here is to construct a countable collection of continuous functions $f_n : X \to [0,1]$ using Urysohn's lemma.

We start by applying Urysohn's lemma with respect to the basis elements. Since X is regular, whenever $x \in B_n$ there is another basis element B_m such that $x \in B_m$ and $\overline{B_m} \subset B_n$. Whenever we have two basis elements satisfying $\overline{B_m} \subset B_n$, we may use Urysohn's lemma to construct a function $g_{n,m} : X \to [0,1]$ so that $g_{n,m}(x) = 0$ for $x \in B_m$ and $g_{n,m}(x) = 1$ for $x \in {B_n}^{\mathsf{c}}$. The collection of all such $g_{n,m}$ is countable, so we may relabel it to produce a sequence of functions $f_n : X \to [0,1]$.

We then define a map $F : X \to Q$ by setting $F(x) = (f_1(x), f_2(x), \ldots)$. It remains to prove that F is an embedding into the product topology on Q.

The map F is certainly continuous, as each of its component functions is continuous. We prove that it is injective. Let $x \neq y$, and pick first a basis element B_n such that $x \in B_n$ but $y \notin B_n$ (using the fact that X is Hausdorff). Then pick B_m such that $x \in B_m$ and $\overline{B_m} \subset B_n$. For the appropriate function $g_{n,m}$ constructed before, we would have $g_{n,m}(x) = 0$ but $g_{n,m}(y) = 1$. If N is such that $f_N = g_{n,m}$, then this implies $f_N(x) \neq f_N(y)$ and therefore $F(x) \neq F(y)$. Thus F is injective.

We need now only show that F is a homeomorphism onto its image, which is done easiest by showing that it is an open mapping into the relative topology on the image set $F(X)$. Let $U \subset X$ be open. We need to show that for each $z \in F(U)$ there is an open neighbourhood V such that $z \in V \subset F(U)$.

Pick $z' \in U$ as the unique point such that $F(z') = z$. By choosing appropriate basis elements such that $z' \in B_m$ and $\overline{B_m} \subset B_n \subset U$, we may find an index N such that $f_N(z') = 0$ and $f_N \equiv 1$ outside of U. Let then $V_0 = \mathrm{proj}_N^{-1}([0,1))$, so that V_0 is an open set in the Hilbert cube Q due to the continuity of the projection map $\mathrm{proj}_N : Q \to [0,1]$. We finally let $V = V_0 \cap F(X)$.

It remains to show that $z \in V \subset F(U)$. Since $f_N(z) = 0$, we find that $\mathrm{proj}_N(f_N(z)) = 0 \in (0,1]$ and hence $z \in V$. Moreover, if $p \notin F(U)$ then by injectivity $F^{-1}(p) \notin U$. Since $f_N \equiv 1$ outside of U, we find that $\mathrm{proj}_N(p) = 1$ and therefore $p \notin V$. This proves that $V \subset F(U)$ and hence the theorem is proven. □

Due to Theorem 4.4, we can also conclude that second-countable spaces which are locally compact and Hausdorff are metrizable.

4.5 The Tietze extension theorem

Another consequence of Urysohn's lemma is the Tietze extension theorem.

> **Theorem 4.11** (Tietze extension theorem)**.** *Let X be normal and $A \subset X$ be closed. Then any continuous function $f : A \to \mathbb{R}$ may be extended to a continuous function $F : X \to \mathbb{R}$. Moreover, if f is bounded then F can be chosen to be bounded as well.*

Proof. The proof is a fairly simple approximation idea using Urysohn's lemma, but we left it in the appendix for the interested reader. See Case A.2. □

4.6 Exercises

Exercise 4.1. Show that a subspace of a regular space is regular.

Exercise 4.2. Show that a closed subspace of a normal space is normal.

Exercise 4.3. Let X be a space equipped with the discrete topology. Is X necessarily

- normal?
- first-countable?
- metrizable?

Exercise 4.4. Show that a space X is normal if and only if for every closed set F and open set U such that $F \subset U$, there is an open set V such that $F \subset \bar{V} \subset U$.

Exercise 4.5. Show that if X is normal and $F, G \subset X$ are closed, then there are open neighbourhoods U, V of F, G respectively such that $\bar{U} \cap \bar{V} = \emptyset$.

Exercise 4.6. Show that the product of regular spaces is regular.

Exercise 4.7. Show that every compact Hausdorff space is normal.

Exercise 4.8. Show that every locally compact Hausdorff space is regular. (Hint: **??** will be useful here.)

Exercise 4.9. Show that every regular Lindelöf space is normal.

Exercise 4.10. A space X is said to be **completely normal** if every subspace of X is normal. Prove that X is completely normal if and only if for every pair of sets $A, B \in X$ with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, there exist disjoint neighbourhoods of A and B.

Exercise 4.11. Are metrizable spaces completely normal?

Exercise 4.12. Let $p : X \to Y$ be closed, continuous, surjective, and such that $p^{-1}(\{y\})$ is compact for every $y \in Y$.

- Show that if X is Hausdorff, then so is Y.
- Show that if X is regular, then so is Y.
- Show that if X is locally compact, then so is Y.
- Show that if X is second-countable, then so is Y.

Exercise 4.13. Show that a set $A \subset X$ is dense if and only if $\mathrm{int}(A^{\mathsf{c}}) = \emptyset$.

Exercise 4.14. Let X be second-countable. Show that if $A \subset X$ is uncountable, then $\mathrm{acc}(A) \cap A$ is uncountable.

Exercise 4.15. Let X be second-countable. Show that if $A \subset X$ is a discrete subspace (meaning that the relative topology on A is the discrete one), then A is countable.

Exercise 4.16. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X with $\mathcal{T} \subset \mathcal{T}'$. Prove that if $(X, \mathcal{T}')$ is separable, then $(X, \mathcal{T})$ is separable.

Exercise 4.17. Let $S \subset X$ be dense and $f, g : X \to Y$ be continuous functions. Suppose that $f(s) = g(s)$ for every $s \in S$. Show that $f(x) = g(x)$ for all $x \in X$.

Exercise 4.18. Let $A_i \subset X_i$ be dense sets for a collection of sets indexed by $i \in I$. Show that $\prod_i A_i$ is dense in the set $\prod_i X_i$.

Exercise 4.19. Show that a closed subspace of a Lindelöf space is Lindelöf.

Exercise 4.20. Show that if $f : X \to Y$ is a continuous surjection, then:

- If X is separable, Y is separable.
- If X is Lindelöf, then Y is Lindelöf.

Exercise 4.21. Suppose that X is locally compact, Hausdorff, and second-countable. Show that there exists a sequence of open sets (U_n) such that $\overline{U_n} \subset U_{n+1}$ for all n and $X = \bigcup_n U_n$.

Exercise 4.22. Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

Exercise 4.23. Show that the Sorgenfrey plane $\mathbb{R}_\ell \times \mathbb{R}_\ell$ introduced in Example 4.1 is not Lindelöf. This shows that the product of two Lindelöf spaces may not be Lindelöf.

Exercise 4.24. Prove that $C([0,1])$, i.e. the set of continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the sup-norm, is a separable space.

Exercise 4.25. Prove that the set of bounded functions on $[0,1]$, equipped with the sup-norm, is not separable.

Exercise 4.26. Let f be the function defined in the proof of Urysohn's lemma. Find an expression for $f^{-1}(\{t\})$ for any $t \in (0,1)$.

Exercise 4.27. Give an example of a second-countable Hausdorff space which is not metrizable.

Exercise 4.28. Show that a locally compact second-countable Hausdorff space is metrizable.

Exercise 4.29. A space X is **locally metrizable** if each point has a neighbourhood which is metrizable as a subspace of X. Show that a locally metrizable compact Hausdorff space is metrizable.

Exercise 4.30. Show that a locally metrizable regular Lindelöf space is metrizable.

Exercise 4.31. Let $f : \mathbb{R} \to \mathbb{R}$ be **continuous from the right** if

$$\lim_{n\to\infty} f(x_n) = f(x) \text{ whenever } x_n \to x \text{ and } x_n > x \text{ for all } n.$$

Recall that $\mathbb{R}_\ell$ denotes the real line equipped with the lower limit topology from Example 4.1. Show that $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous if it is continuous from the right.

Exercise 4.32. Is the topology of pointwise convergence on the set $L^\infty([0,1])$ (bounded functions from $[0,1]$ to $\mathbb{R}$) metrizable?

5 Special spaces and results

5.1 Quotient topology

In this subsection we introduce *quotient topology*, which is a way to construct a new topological space based on a given topological space X and an equivalence relation $\sim$ on X. Recall that an equivalence relation $\sim : X \times X \to \{0, 1\}$ is a binary relation which practically just lets us decide whether two elements $a, b \in X$ are 'equivalent' or not, denoted by $a \sim b$ if they are equivalent under $\sim$. Equivalence relations satisfy the following three axioms:

- (Reflexivity) $a \sim a$ for all $a \in X$.
- (Symmetry) If $a \sim b$, then $b \sim a$.
- (Transitivity) If $a \sim b$ and $b \sim c$, then $a \sim c$.

While equivalence relations are a fairly abstract concept, there is a nice geometric interpretation here. Let X be some shape, and $\sim$ be an equivalence relation on X. Then $\sim$ gives us a way to create a new shape out of X by gluing together parts of X: we glue two points together if and only if they are equivalent under $\sim$. For example, a torus can be created out of a sheet of paper using this idea:

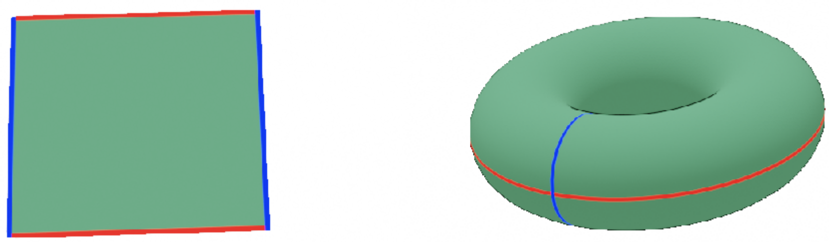


Figure 11: Creating a torus from a sheet of paper by gluing the colored edges together. This gluing can be given as an equivalence relation on $[0, 1]^2$ where each point on the blue sides is equivalent with a corresponding point on the other blue side, and vice versa for the red sides. The four corners are also all equivalent.

In this torus example, the square $[0, 1]^2$ is naturally a topological space as a subspace of $\mathbb{R}^2$. But the torus which is created here is, intuitively, a completely different topological object than the original square. This begs the question: How should we define a topology on the torus which respects this gluing process?

First, let's make this process mathematically precise. Given a space X and an equivalence relation $\sim$, we let $X/\sim$ denote the **quotient space** of X w.r.t. $\sim$, meaning the space of *equivalence classes* in the relation $\sim$. Equivalence classes are simply the parts of X in which each point is equivalent to each other: each point $x \in X$ belongs to some equivalence class $E_x = \{y \in X : x \sim y\}$, and these sets together form a disjoint partition of X.

There is a natural surjection $p : X \to X/\sim$ defined by setting $p(x) = E_x$, i.e. each point is mapped to its own equivalence class.

Definition 5.1. Let X be a topological space and $\sim$ be an equivalence relation.

Then the **quotient topology** on $X/\sim$ is defined as the topology induced by p. Hence a set $U \subset X/\sim$ is open in the quotient topology if and only if $p^{-1}(U)$ is open in X.

In other words, a subset of $X/\sim$ (which is a collection of sets of X), is open if and only if the union of its elements is an open set in X.

Another way to think about quotient topology is simply in terms of the *quotient map* $p : X \to Y$.

Definition 5.2. A map $p : X \to Y$ between topological spaces is a **quotient map** if it is surjective and the following condition holds: A set $V \subset Y$ is open if and only if $p^{-1}(V)$ is open.

If Y has no topology yet and a surjective map $p : X \to Y$ is given, then the topology induced by p on Y is precisely the topology in which p becomes a quotient map. The equivalence relation related to this map is given by $x \sim y \Leftrightarrow p(x) = p(y)$.

Example 5.1. The quotient space of a Hausdorff space does not need to be Hausdorff. Let us take two copies of $\mathbb{R}$, say $X = (\mathbb{R} \times \{a\}) \cup (\mathbb{R} \times \{b\})$. This space is certainly Hausdorff. Define an equivalence relation $\sim$ on X by identifying points (x, a) and (x, b) together whenever $x \neq 0$.

The quotient space $X/\sim$ is called the **line with two origins**, and it is not Hausdorff because the two 'origins' $(0, a)$ and $(0, b)$ cannot be separated. Indeed, any neighbourhood of $(0, a)$ or $(0, b)$ must contain points (ϵ, a) and (ϵ, b) for small numbers ϵ and therefore any two such neighbourhoods will intersect.

5.2 Topological vector spaces

Especially in functional analysis, there is special interest towards studying vector spaces which are also topological spaces. The topology in these spaces is not necessarily given by a norm, but rather one may end up working with *topological vector spaces* where the relation between the vector space structure and the topology is more subtle. Nevertheless, many interesting properties still hold in such spaces.

Definition 5.3. A vector space X over the scalar field $\mathbb{R}$ is a **topological vector space** if it is equipped with a topology where

- Vector addition $(x, y) \mapsto x + y$ defines a continuous map from $X \times X$ to X.
- Scalar multiplication $(\lambda, x) \mapsto \lambda x$ defines a continuous map from $\mathbb{R} \times X$ to X.

The scalar field $\mathbb{R}$ here may be replaced by any other field such as the complex numbers $\mathbb{C}$, but we work with $\mathbb{R}$ here for simplicity.

Note that the assumptions about addition and scalar multiplication being continuous *from the product spaces* are fairly strong and useful assumptions. The basic consequence is that the translation map $x \mapsto x + x_0$ and the dilation map $x \mapsto \lambda x$ are continuous maps for all $x_0 \in X$ and $\lambda \in \mathbb{R}$ (see also Theorem 5.3), but there are some further major consequences which we will explore here.

Note!

Some notation before we proceed. We denote by $\mathcal{A} : X \times X \to X$ the addition map $(x, y) \mapsto x + y$. We also define addition for subsets of X as follows. If $U, V \subset X$, then

$$U + V := \mathcal{A}(U \times V) = \{x + y : x \in U, y \in V\}.$$

We can similarly define $\lambda U = \{\lambda u : u \in U\}$ for any scalar λ and $x + U := \{x + u : u \in U\}$ for any $x \in X$.

Next we explore further interaction between the vector space structure and the topology on X.

Theorem 5.1. *Let X be a topological vector space. Then the following hold:*

- *If $U \subset X$ is open, then $U + A$ is open for any $A \subset X$.*
- *If $x \in X$ and $A \subset X$, then $\overline{x + A} = x + \overline{A}$.*
- *If $A, B \subset X$, then $\overline{A} + \overline{B} \subset \overline{A + B}$.*
- *If $F, G \subset X$ are compact, then $F + G$ is compact.*
- *If $F \subset X$ is compact and $G \subset X$ is closed, then $F + G$ is closed.*

The continuity of the addition map can be alternatively characterized as follows.

Theorem 5.2. *The addition map $\mathcal{A}$ is continuous at $(0,0)$ if and only if for each neighbourhood $U \subset X$ of 0, there is another neighbourhood V of 0 such that $V + V \subset U$.*

The harder direction can be proven simply by using the continuity of $\mathcal{A}$ to take a basis element $B \times C$ containing $(0,0)$ such that $\mathcal{A}(B \times C) = B + C \subset U$, and letting $V = B \cap C$.

The following result shows that the topology in a topological vector space is translation- and scaling-invariant.

Theorem 5.3. *Let X be a topological vector space, $x_0 \in X$, and $\lambda \in \mathbb{R} \setminus \{0\}$. Then both the translation map $x \mapsto x + x_0$ and the dilation map $x \mapsto \lambda x$ are homeomorphisms of X with itself.*

To check whether a topological vector space is Hausdorff, it is enough to check the weaker condition of singletons being closed sets:

Theorem 5.4. *A topological vector space is Hausdorff if and only if it is T_1.*

Due to Theorem 5.3, for the Hausdorff property it's actually enough just to check that $\{0\}$ is a closed set in X.

Proof. Suppose singletons are closed. Let $x \neq y$ and we wish to find disjoint neighbourhoods of x, y. Since $x - y \neq 0$, there is a neighbourhood U of 0 such that $x - y \neq U$ since 0 is in the exterior of $\{x - y\}$.

Use Theorem 5.2 to pick a neighbourhood V of 0 such that $V + V \subset U$. Now let

$$A = x - V \qquad B = y + V,$$

and we claim that A and B are the disjoint neighbourhoods we are after. If there was a point $z \in A \cap B$, then $z = x - v = y + w$ for some $v, w \in V$. Thus $x - y = v + w \in V + V \subset U$, which is a contradiction. □

5.3 Weak topology

As before, X denotes a topological vector space over $\mathbb{R}$ here. The definitions and results are the same for any other scalar field $\mathbb{K}$ in place of $\mathbb{R}$.

The concept of a *weak topology* is very important for applications when studying objects living in topological vector spaces. For example, in the study of function spaces it is common to work with a sequence $(f_n) \subset X$ which does not (a priori) converge in the conventional topology on X, but may still have a limit a weaker sense. To define the weak topology we introduce *continuous linear functionals* on X.

Definition 5.4. If a function $F : X \to \mathbb{R}$ is continuous and linear then we call F a **continuous linear functional** on X.

The space of all continuous linear functionals on X is denoted by X^* and is called the **dual space** of X.

There are some interesting characterizations of when a functional $F : X \to \mathbb{R}$ is continuous:

Theorem 5.5. *Let $F : X \to \mathbb{R}$ be a linear functional which is not identically zero. Then the following are equivalent:*

- *F is continuous.*
- *The **kernel** $\ker(F) := F^{-1}(\{0\}) = \{x \in X : F(x) = 0\}$ is a closed set.*
- *The set $\ker(F)$ is not dense in X.*
- *There exists a neighbourhood of 0 in which F is bounded.*

Proof. If F is continuous, then all of the three other points follow easily: The kernel is closed because it is the preimage of a closed set. It is not dense because a closed dense subset would be the whole space and then we would have $F \equiv 0$ which was not allowed. And $U = F^{-1}((-1, 1))$ is a neighbourhood of 0 in X in which F is certainly bounded.

Let us next start by proving that if F is bounded in a neighbourhood U of 0, then it is continuous. Note that it is enough to only prove continuity at 0, since then we can get continuity at every other point by translation (since translation is a homeomorphism of X). To prove this, pick $\epsilon > 0$ and consider the neighbourhood $(-\epsilon, \epsilon)$ of 0 in $\mathbb{R}$. Then, since $F(U) \subset (-M, M)$ for some $M > 0$, we have that

$$F\left(\frac{\epsilon}{M}U\right) \subset (-\epsilon, \epsilon),$$

which proves continuity at 0.

Let us then assume that $\ker(F)$ is closed. Since $\ker(F) \neq X$, there is some $x \notin \ker(F)$. Let U be a neighbourhood of x which does not intersect $\ker(F)$, in which case $x - U$ is a neighbourhood of 0 in which F does not attain the value $F(x)$. One can show (Exercise 5.13) that there is a neighbourhood V of 0 such that $V \subset x - U$ and $\lambda V \subset V$ whenever $|\lambda| \leq 1$. Suppose that F is not bounded in V. Then there must be an $y \in V$ such that $F(y) > |F(x)|$. But now if

$$\lambda = \frac{F(x)}{F(y)}, \qquad \text{then} \qquad F(\lambda y) = \lambda F(y) = F(x),$$

however, $\lambda y \in V$ since $|\lambda| < 1$. This is a contradiction since F was not supposed to attain the value $F(x)$ at a point $\lambda y \in U$, so F is bounded and therefore continuous.

Suppose now that $\ker(F)$ is not dense. Then there is again an open set U in X which does not intersect $\ker(F)$. We can therefore argue the same way as in the previous point to show that F is bounded and thus continuous. □

We next define the weak topology on X.

Definition 5.5. The weak topology on X is the topology induced by the functions in the dual space X^*, i.e. the coarsest topology under which every continuous linear functional (in the original topology) $F : X \to \mathbb{R}$ is continuous.

One might wonder whether the weak topology truly is weaker than the original topology. Certainly the original topology contains the weak topology since the functionals in X^* are continuous under it, but these topologies are not necessarily equal.

Example 5.2. Let $X = \ell^2$, the space of all square-summable sequences of real numbers. This is a Hilbert space with the inner product

$$\langle x, y \rangle = \left(\sum_{k=1}^{\infty} x_k y_k\right)^{1/2}.$$

Since X is a Hilbert space, it is its own dual, meaning that for every linear functional $F \in X^*$ there is a $y \in X$ such that

$$F(x) = \langle x, y \rangle \qquad \text{for all } x \in X.$$

Let now $e^{(n)} \in X$ denote the sequence with $e^{(n)}_k = 1$ if $k = n$ and $e^{(n)} = 0$ otherwise. Then the sequence $(e^{(n)})$ does not converge in the norm topology.

However, it does converge in the weak topology. Given any $y \in X$ representing a dual element, the sequence $\langle e^{(n)}, y \rangle = y_n$ converges to zero as $n \to \infty$ since $||y||_2 = \sum_n |y_n|^2 < \infty$. By Theorem 2.10, this shows that $e^{(n)}$ converges to the constant sequence of zeroes in the weak topology.

The dual space X^* has a topology given by the duality pairing called the *weak*-topology*. It turns out that there is a way to identify each point $x \in X$ with a continuous linear functional on X^*, provided by the map $\chi_x : X^* \to \mathbb{R}$ defined by

$$\chi_x(F) = F(x).$$

We use these maps to induce a topology on the dual space.

Definition 5.6. The weak*-topology on X^* is defined as the topology induced by the collection of maps $\{\chi_x : x \in X\}$.

If X was also a normed space with norm $||\cdot||_X$, then the dual space X^* itself is a normed space under the **operator norm** defined by

$$||f|| = \sup_{||x||_X \leq 1} |f(x)|. \tag{5.1}$$

This topology is finer than the weak*-topology.

5.4 Tychonoff's theorem

In this section we briefly mention a profound theorem in topology with many important consequences in other areas of mathematics.

Theorem 5.6 (Tychonoff's theorem)**.** *The product of compact topological spaces is compact.*

Though this theorem looks fairly innocent, its proof is quite involved due to the fact that the result becomes highly nontrivial for infinite products of compact spaces. We will omit the full proof here, but show that Tychonoff's theorem is true for finite products instead to give a taste of the result:

Proof. Let us prove that if X and Y are compact, then $X \times Y$ is compact. Hence, suppose that $X \times Y$ has an open cover $\mathcal{D}$.

For each $x \in X$, note that $\mathcal{D}$ is also an open cover of $\{x\} \times Y$. Since $\{x\} \times Y$ is homeomorphic to Y, it is compact. Hence there is a finite subset $\mathcal{D}_x \subset \mathcal{D}$ which covers $\{x\} \times Y$. Let now

$$U_x = \bigcap \{\mathrm{proj}_X(D) : D \in \mathcal{D}_x \text{ and } D \cap (\{x\} \times Y) \neq \emptyset\}.$$

This is an open set in X since it is the finite intersection of open sets (recall also that the projection map is an open map, see Exercise 2.36). It also contains the point x since $\mathcal{D}_x$ covers $\{x\} \times Y$ and we only included sets in the intersection which actually intersect $\{x\} \times Y$. Thus U_x is a neighbourhood of x. The set $\{U_x : x \in X\}$ is therefore an open cover of X, so there is a finite subcover $U_{x_1}, \ldots, U_{x_n}$ with some points $x_1, \ldots, x_n \in X$ by compactness of X.

Now the set $\mathcal{D}_{x_1} \cup \cdots \cup \mathcal{D}_{x_n}$ will be a cover of $X \times Y$ and hence a finite subcover of $\mathcal{D}$. The reasoning why it covers $X \times Y$ is that if we want to find the element which covers any point $(x, y) \in X \times Y$, we first find the appropriate U_{x_j} which contains x and then there must be a member of $\mathcal{D}_{x_j}$ which contains (x, y) since $\mathcal{D}_{x_j}$ must be a cover of $\{x\} \times Y$ as well. ☐

5.4.1 The Banach-Alaoglu theorem

One major consequence of Tychonoff's theorem is the Banach-Alaoglu theorem which has wide applications in functional analysis.

Theorem 5.7 (Banach-Alaoglu theorem)**.** *Let X be a normed topological vector space. Then the closed unit ball of the dual space X^* is compact in the weak*-topology.*

Proof. We first shed some light into what the theorem statement actually means. The key players here are the dual space X^*, consisting of continuous linear functionals $F : X \to \mathbb{R}$, its closed unit ball (in the operator norm, Equation (5.1))

$$B^* = \{F \in X^* : ||F|| \leq 1\},$$

and the weak*-topology on X^* which was defined as the topology induced by the maps $\chi_x : X^* \to \mathbb{R}$ given by $\chi_x(F) = F(x)$ for each $x \in X$. Hence we are to prove that B^* is compact in this induced topology (rather than the topology given by the operator norm that was used to define it).

The key idea here is that there is a natural correspondence between **functions** $F : X \to \mathbb{R}$ and **elements of the product space** $\mathbb{R}^X = \prod_X \mathbb{R}$. Indeed, any element of the product space $\mathbb{R}^X$ is, by definition (see the discussion in Section 2.6), just a function from X to $\mathbb{R}$. Moreover, the components of an element $F \in \mathbb{R}^X$ are, again by definition, the values $F(x)$ for each $x \in X$.

This identification lets us identify the dual space X^* as a subset of the product space $\mathbb{R}^X$, since the elements in X^* are precisely the elements of $\mathbb{R}^X$ which correspond to functions that are continuous and linear. Moreover, for each $F \in \mathbb{R}^X$ the value $F(x)$ is equal to both the component of F as an element of the product space, aka $\text{proj}_x(F)$, as well as $\chi_x(F)$ by definition. Since the projection maps proj_x are used to induce the product topology on $\mathbb{R}^X$, we find that the weak topology of the dual space X^* induced by the maps χ_x is the same topology as the relative topology of X^* as a subspace of $\mathbb{R}^X$. This completes our identification, as we now only need to show that the closed unit ball B^*, identified with a subset of the product space $\mathbb{R}^X$, is compact in the product topology.

Let us therefore try to express the closed unit ball $B^* = \{F \in X^* : ||F|| \leq 1\}$ in product space terms. The fact that the operator norm of $F : X \to \mathbb{R}$ is at most one means exactly that $|F(x)| \leq ||x||_X$ for all $x \in X$. Hence if we define the intervals $I_x = [-||x||_X, ||x||_X]$ on $\mathbb{R}$, we can consider any functional $F \in B^*$ as an element of the product set $\mathcal{I} = \prod_{x \in X} I_x$. Hence

$$\mathcal{B}^* = \{F \in \mathcal{I} : F \text{ represents a linear function } X \to \mathbb{R}\}.$$

The main point here is that the set $\mathcal{I}$ is compact by Tychonoff's theorem because it is the product of the compact sets I_x. We would like to conclude that B^* is compact by showing that it is a closed subset of $\mathcal{I}$.

The reason why B^* is not equal to the entire set $\mathcal{I}$ is because B^* is a subset of X^* and therefore contains only those kinds of elements $F \in \mathbb{R}^X$ which correspond to linear functions from X to $\mathbb{R}$. Now the key trick is to notice that a function $F : X \to \mathbb{R}$ is linear if and only if it satisfies

$$F(x + \lambda y) - F(x) - \lambda F(y) = 0 \qquad \text{for all } x, y \in X \text{ and } \lambda \in \mathbb{R}.$$

In other words, if for each $x, y \in X$ and $\lambda \in \mathbb{R}$ we define the map $T_{x,y,\lambda} : \mathbb{R}^X \to \mathbb{R}$ by setting

$$T_{x,y,\lambda}(F) = F(x + \lambda y) - F(x) - \lambda F(y),$$

then the set of linear functions $L \subset \mathbb{R}^X$ is precisely equal to

$$L = \bigcap_{x,y \in X, \lambda \in \mathbb{R}} T^{-1}_{x,y,\lambda}(\{0\}).$$

This set is closed because the maps $T_{x,y,\lambda}$ are continuous since they are just linear combinations of projection maps, and therefore the preimages $T^{-1}_{x,y,\lambda}(\{0\})$ and their arbitrary intersections are closed as well. Hence the set $B^* = \mathcal{I} \cap L$ is a compact set as the intersection of a compact set with a closed set. □

5.5 Baire spaces

We start this subsection with the following result.

Theorem 5.8 (Baire theorem)**.** *Let X be a locally compact Hausdorff space and $U_n \subset X$ be a countable collection of open dense sets. Then $\bigcap_n U_n$ is dense.*

Proof. We may actually suppose that X is compact since denseness is something that is enough to be checked locally: If a set is dense in some neighbourhood of any point in X, then it is dense in X. Let thus X be compact.

Let V be any open set in X. We wish to show that V contains a point in $\bigcap_n U_n$. We use an inductive argument.

Since U_1 is dense, there is a point $x_1 \in U_1 \cap V$. Since X is compact and Hausdorff, it is regular (see Theorem 4.4), we can then pick a neighbourhood V_2 of x_1 such that $\overline{V_2} \subset U_1 \cap V$. Continue this process by picking an $x_2 \in U_2 \cap V_2$ and a neighbourhood V_3 of x_2 such that $\overline{V_3} \subset U_2 \cap V_2$, and so on.

This gives a sequence of nonempty sets V_n with the property that $\overline{V_{n+1}} \subset V_n$. Hence the closed sets $\overline{V_n}$ have the finite intersection property, so their total intersection $\bigcap_n \overline{V_n}$ is also nonempty by Theorem 3.3. But this total intersection must be a subset of both V and $\bigcap_n U_n$ (because $\overline{V_n} \subset U_n$ for all n), which means that $\bigcap_n U_n$ is dense since V was arbitrary. □

In general, any space with the property that the intersection of a countable collection of open dense sets is still dense is called a **Baire space**.

Note!

An equivalent way to state the condition of a Baire space is by taking complements and using closed sets instead. A space is a Baire space if and only if for each countable collection of closed subsets F_n with empty interior $\operatorname{int} F_n = \emptyset$, the union $\bigcup_n F_n$ also has empty interior.

5.6 Order topology

In this subsection, we introduce a topology defined on **totally ordered sets**. A **total order** on a set X is a function $\leq : X \times X \to \{0, 1\}$ which works just like the usual ordering of real numbers:

- $a \leq a$ for all $a \in X$.
- If $a \leq b$ and $b \leq c$, then $a \leq c$.
- If $a \leq b$ and $b \leq a$, then $a = b$.
- Either $a \leq b$ or $b \leq a$ for all $a, b \in X$.

Furthermore, for a total order $\leq$ we denote that $a < b$ if $a \leq b$ and $a \neq b$.

Definition 5.7. The **order topology** on a totally ordered set X is the topology generated by the basis

$$(a, b) := \{x \in X : a < x < b\}, \qquad \text{for } a, b \in X.$$

The order topology is always normal and Hausdorff.

5.7 Homotopy

Homotopy concerns the idea of finding a *continuous deformation* between two different spaces. It becomes an incredibly useful tool in the classification of topological spaces and objects.

Let us start by defining homotopy between to continuous maps:

Definition 5.8. Let $f, g : X \to Y$ be continuous. Then we say that f and g are **homotopic** if there exists a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such a map H is called a **homotopy** between f and g.

The interpretation here is that the second parameter in the product space $X \times [0, 1]$ describes time, and the homotopy H describes a process in which the map f deforms continuously to the map g. At each time $t \in (0, 1)$, there is an intermediate map $h_t : X \to Y$ given by $h_t(x) = H(x, t)$.

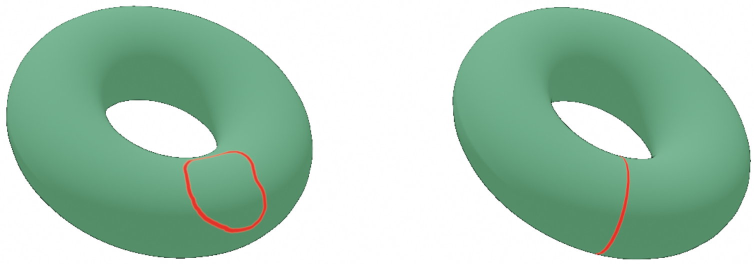


Figure 12: A typical example of maps which are **not** homotopic is given by two paths $f, g : S^1 \to S^1 \times S^1$ on the torus, one which loops around and one which doesn't. Intuitively, the reason for this is that the path on the left can be contracted to a point, while the path on the right that goes around the torus can't.

We next come to the homotopy equivalence between spaces.

Definition 5.9. Two topological spaces X and Y are said to be **homotopy equivalent** if there is a pair of maps $f : X \to Y$ and $g : Y \to X$ such that the map $g \circ f$ is homotopic to id_X and the map $f \circ g$ is homotopic to id_Y.

Spaces which are homotopy equivalent to a single point are called **contractible**.

One may check that a space is contractible if and only if the identity map is homotopic to a constant map.

5.7.1 Isotopy

Let us consider the following two configurations of placing two rings inside $\mathbb{R}^3$:

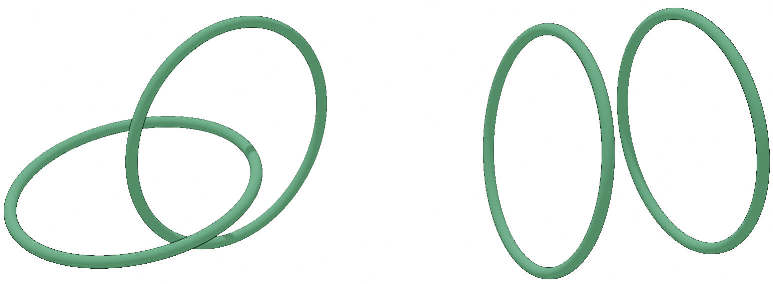


Figure 13: Two ring configurations in $\mathbb{R}^3$ are homeomorphic as subspaces, but not isotopic as embeddings.

Interpreting the configuration on the left as one subset of $\mathbb{R}^3$ and the configuration on the right as another, there is certainly something common about the two. Indeed, these sets are homeomorphic to each other when considered as subspaces in the relative topology. However, there is perhaps something displeasing about this conclusion considering that there is no way to move around the rings on the left to produce the configuration on the right without the rings passing through each other or being torn apart. Even if squeezing and stretching is allowed, this seems an impossibility. The concept of *isotopy* gives us a way to formalize this idea.

Definition 5.10. A homotopy $H(x, t) : X \times [0, 1] \to Y$ is an **isotopy** if, for each fixed t, the map $x \to H(x, t)$ is an embedding.

The main point here being that the deformation remains injective at each time point t.

5.8 Exercises

Exercise 5.1. Let $Y_1, Y_2 \subset Y$ be two disjoint connected components. Let $f_1, f_2 : X \to Y$ be such that $f_1(X) \subset Y_1$ and $f_2(X) \subset Y_2$. Show that f_1 and f_2 are not homotopic.

Exercise 5.2. Prove Tychonoff's theorem in the case of two spaces: Let X and Y be compact. Show that $X \times Y$ is compact.

Exercise 5.3. Show that the line with two origins considered in Example 5.1 is a T_1-space and second-countable.

Exercise 5.4. Show that an open surjective map $p : X \to Y$ is a quotient map.

Exercise 5.5. Show that a closed surjective map $p : X \to Y$ is a quotient map.

Exercise 5.6. Find a quotient map which is neither open nor closed.

Exercise 5.7. Let $X = \mathbb{R}/\mathbb{Q}$ be the quotient space obtained by identifying $x, y \in \mathbb{R}$ if $x - y \in \mathbb{Q}$. Prove that the topology on X is the mini-topology $\mathcal{T}_{mini} = \{\emptyset, X\}$.

Exercise 5.8. Let X be a topological vector space. Prove that if $U \subset X$ is open, then $U + A$ is open for any $A \subset X$.

Exercise 5.9. Let X be a topological vector space. Prove that if $A, B \subset X$, then $\overline{A} + \overline{B} \subset \overline{A + B}$.

Exercise 5.10. Give an example of closed sets $A, B \subset \mathbb{R}^2$ such that $A + B$ is not closed.

Exercise 5.11. Let X be a topological vector space. If $F, G \subset X$ are compact, then $F + G$ is compact.

Exercise 5.12. Let X be a topological vector space. If $F \subset X$ is compact and $G \subset X$ is closed, then $F + G$ is closed.

Exercise 5.13. A subset $A \subset X$ is called **balanced** if $\lambda A \subset A$ whenever $|\lambda| \leq 1$. Prove that for every neighbourhood U of 0, there is a balanced neighbourhood V of 0 such that $V \subset U$.

Hint: Use the continuity of scalar multiplication $\mathcal{M} : \mathbb{R} \times X \to X$.

Exercise 5.14. Let $X = \ell^2$. Is the open unit ball $B(0, 1)$ in the norm topology on X also an open set in the weak topology?

Exercise 5.15. Let X be a normed space, in which case X^* is also a topological vector space with the topology given by the operator norm. Which topology is finer on X^*, the weak*-topology or the weak topology (of X^*, not X)?

A Appendix

A.1 Omitted proofs

In this subsection we provide proofs for some things which were not fully explained in the main text.

Case A.1. Here we give explanation for the statements in Example 4.1. First of all, let us prove that any subset of Δ is closed. Given a subset $A \subset \Delta$, we consider its complement $\mathbb{R}^2_\ell \setminus A$. We think of this complement in two parts: The part below Δ: $\{(x, y) : y < -x\}$, which is open because it is open in the usual topology on $\mathbb{R}^2$ and the topology on the Sorgenfrey plane is finer, and the other part which can be written as the union of basis elements $[a, b) \times [c, d)$ with $(a, c) \in \Delta \setminus A$ or $c > -a$. Since the complement of A is then an union of open sets, it is open.

We next recall the set $F = \{(x, -x) : x \in \mathbb{Q}\}$, and prove that there are no two neighbourhoods of F and $\Delta \setminus F = \{(x, -x) : x \notin \mathbb{Q}\}$ which are disjoint. Suppose that U is a neighbourhood of $\Delta \setminus F$.

For each $n \in \mathbb{Z}_+$, let $S_n \subset [0, 1]$ denote the set of points $x \in [0, 1]$ such that $[x, x+1/n)\times[-x, -x+1/n) \subset U$. Note that for each $x \in [0, 1] \setminus \mathbb{Q}$, there must be some n_x such that $x \in S_{n_x}$ because x must be in some basis element contained in U.

We would like to show that there is some n such that $\overline{S_n}$ (here the closure taken in the usual topology on $[0, 1]$) contains an open interval (a, b). Suppose to the contrary that all of the sets $\overline{S_n}$ have empty interior in $[0, 1]$. The collection $\{S_n : n \in \mathbb{Z}_+\}$ is a countable collection of closed sets with empty interiors. Another such collection is $\{\{q\} : q \in [0, 1] \cap \mathbb{Q}\}$. We can combine these two collections into yet another collection with the same properties (countable, closed sets, empty interiors).

Now since $[0, 1]$ is compact and Hausdorff, Theorem 5.8 and the note after imply that also the set

$$\bigcup_n \overline{S_n} \ \cup \ ([0, 1] \cap \mathbb{Q})$$

has empty interior. But if $x \in [0, 1]$, then either $x \in \mathbb{Q}$ or $x \in S_{n_x}$ for some n_x, so the union above is the whole interval $[0, 1]$. This is a contradiction, so some $\overline{S_n}$ has nonempty interior and contains an interval (a, b).

We now claim that for any point $q \in (a, b) \cap \mathbb{Q}$, the point $(q, -q) \in F$ belongs to $\bar{U}$, where U was the neighbourhood of $\Delta \setminus F$. To prove this, it is enough to show that any basis element of the form $[q, q + \epsilon) \times [-q, -q + \epsilon)$ with $\epsilon > 0$ intersects U. To prove this, we pick $t \in (a, b) \setminus \mathbb{Q}$ so that $|q - t| < \min(\epsilon, 1/n)/2$. Recall that $[t, t + 1/n) \times [-t, -t + 1/n) \subset U$ since $(a, b) \subset S_n$. By simple geometry it is now possible to see that the sets

$$[q, q + \epsilon) \times [-q, -q + \epsilon) \quad \text{and} \quad [t, t + 1/n) \times [-t, -t + 1/n)$$

must intersect at some point which then lies in U. Thus $(q, -q) \in \bar{U}$, and therefore there can't be an open neighbourhood of F which does not meet the neighbourhood U of $\Delta \setminus F$. Hence $\mathbb{R}^2_\ell$ is not normal.

Example A.1. We try to construct here an example of a sequence (x_n) in a topological space X so that there is a point a which is a cluster point of the sequence (x_n), meaning that every neighbourhood of a contains infinitely many points in (x_n), but no subsequence of (x_n) converges to a.

Let us create this example constructively. We choose $X = \mathbb{N} \cup \{a\}$ for some $a \notin \mathbb{N}$, and aim to pick the topology on X so that the sequence $x_n = n, n \in \mathbb{N}$ will satisfy the requirements above.

For the topology on X, we choose a topology generated by a basis consisting of first the sets $\{n\}$ for each $n \in \mathbb{N}$, and secondly some collection of sets $U \cup \{a\}$ where $U \in \mathcal{U}$ is chosen from some collection $\mathcal{U}$ of subsets of $\mathbb{N}$.

Choosing this collection $\mathcal{U}$ turns out to be hard. The condition of every neighbourhood of a containing infinitely many points in (x_n) is very easy to satisfy by having every member $U \in \mathcal{U}$ be an infinite set. But is is much harder to guarantee that no subsequence of $\{0, 1, 2, \ldots\}$ converges to a.

The heavy machinery we need here is the existence of **free ultrafilters**. The ultrafilter lemma says that, in particular, there is a collection of subsets of $\mathbb{N}$ which is a **filter**, i.e. it satisfies:

(1) $\mathbb{N} \in \mathcal{U}$, but $\emptyset \notin \mathcal{U}$.

(2) If $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$.

(3) If $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$.

Moreover, one can find a filter which is also a free ultrafilter, meaning that:

(1) There is no filter $\mathcal{U}^*$ for which $\mathcal{U} \subset \mathcal{U}^*$.

(2) There is no $x \in \mathbb{N}$ such that $x \in \mathcal{U}$ for all $U \in \mathcal{U}$, i.e. $\bigcap_{U \in \mathcal{U}} U = \emptyset$.

Ultrafilters $\mathcal{U}$ also have the following special property due to their maximality (fourth condition above):

$$\text{If } A \subset U \in \mathcal{U}, \text{ then either } A \in \mathcal{U} \text{ or } U \setminus A \in \mathcal{U}. \tag{A.1}$$

Picking a free ultrafilter $\mathcal{U}$ to construct our topology on $\mathbb{N}$ as described above, we find that each element in $\mathcal{U}$ is infinite since otherwise the combination of the second and fifth condition above would lead to a contradiction. On the other hand, if we consider a subsequence representing a set $A \subset \mathbb{N}$, then we have two cases.

If $A \notin \mathcal{U}$, then we must have $\mathbb{N} \setminus A \in \mathcal{U}$. Then $\{a\} \cup \mathbb{N} \setminus A$ is a neighbourhood of a containing no points in the subsequence represented by A.

If $A \in \mathcal{U}$, then split A into two disjoint infinite subsets A_1, A_2. Exactly one of these must be in $\mathcal{U}$, let this one be A_1. Then $\{a\} \cup A_1$ is a neighbourhood of a but infinitely many members of the subsequence represented by A are not contained inside this neighbourhood.

Thus this subsequence cannot converge to a, and since it was arbitrary we have shown that no subsequence of $\{0, 1, 2, \ldots\}$ converges to a.

Note that the space X constructed here is Hausdorff. Indeed, every point $n \in \mathbb{N}$ has its own neighbourhood $\{n\}$, while the point a may be separated from any such n because the set $\mathbb{N} \setminus \{n\}$ must be in $\mathcal{U}$ due to (A.1).

Example A.2. There is a space which is compact but not sequentially compact. Let S denote the set of strictly increasing sequences of natural numbers, and let $X = \{0, 1\}^S$. Then X is compact due to Tychonoff's theorem (Theorem 5.6). However, X is not sequentially compact as we will now show.

To pick a sequence $(a^{(n)})$ in X for which no subsequence converges, for each $n \in \mathbb{N}$ and $s \in S$ (since s is a sequence, we may write it as $s = (s_k)$ we let $a_s^{(n)} = \text{proj}_s(a^{(n)}) = 1$ if $n = s_k$ for some even number k and $a_s^{(n)} = 0$ otherwise.

Suppose now that there is a subsequence $(a^{(n_k)})$ which converges. Then the sequence (n_k) itself represents a point $s \in S$. However, the components $a_s^{(n_k)}$ cannot converge, because for even numbers k we have $a_s^{(n_k)} = 1$ and for odd numbers k we have $a_s^{(n_k)} = 0$. This is a contradiction so X is not sequentially compact.

Example A.3. There is a space which is sequentially compact but not compact. Let $X = \{0, 1\}^{\mathbb{R}}$ (this is homeomorphic to the space X in the previous example). Let $Y \subset X$ be the subset of elements a for which $a_x = 1$ for countably many $x \in \mathbb{R}$.

Recall that X is compact due to Tychonoff's theorem (Theorem 5.6), so Y is compact only if it is closed in X. But it is not closed, because even every basis element in the product topology on X intersects Y so Y is dense in X but strictly smaller than X.

However, Y is sequentially compact. Indeed, if $(a^{(n)})$ is a sequence in Y then we can find a subsequence which converges as follows. We let $A \subset \mathbb{R}$ denote the subset of points where $a_x^{(n)} = 1$ for some $x \in A_N$ and some n. Then A is a countable set as the countable union of countable sets, so we represent it as $x_1, x_2, \ldots$.

Now we start constructing a countable number of subsequences and finish with a diagonal argument. For the first subsequence $(a^{(n),1})$, if there are finitely many n such that $a_{x_1}^{(n)} = 1$, then we discard those and let the remaining sequence be $(a^{(n),1})$. If there are infinitely many n, we discard all the other n and let $(a^{(n),1})$ be the subsequence that remains.

We continue this process inductively, defining $(a^{(n),k})$ as a subsequence of $(a^{(n),k-1})$ so that the component at x_k for this k:th subsequence is either always 0 or always 1.

If this process finishes (A was finite), then the last remaining subsequence converges. If not, the diagonal sequence $(a^{(k),k})$ converges. In either cases the limit point is a point $a \in Y$ such that $a_x = 1$ for x in some subset $A' \subset A$, and $a_x = 0$ otherwise. Therefore Y is sequentially compact.

Case A.2. Proof of the Tietze extension theorem Theorem 4.11.

First of all, we may assume that f takes values in the interval $[-1,1]$. If this were not the case, we may use the fact that $\mathbb{R}$ is homeomorphic with $(-1,1)$ to make a change of variables on the target side and replace the problem of extending a map $f : A \to \mathbb{R}$ with a problem of extending a map $g : A \to [-1,1]$ instead (here g is obtained from f via composition with a homeomorphism of $\mathbb{R}$ to $(-1,1)$ and vice versa).

Let hence $f : A \to [-1,1]$. The idea now is to find a sequence of functions $H_n : X \to [-1,1]$ defined on the whole space which approximate f uniformly on A. To achieve this, the key idea is to first prove the following claim.

Claim. If $g : A \to [-a,a]$ is continuous, then there exists a continuous $h : X \to [-a/3, a/3]$ such that $|g(x) - h(x)| \le 2a/3$ for all $x \in A$.

Proof of claim. Let us pick sets $A' = g^{-1}([-a,-a/3])$ and $B' = g^{-1}([a/3,a])$, which are closed sets since they are the preimages of closed sets under a continuous map g. Since they are disjoint and X is normal, Urysohn's lemma tells us that there is a function $h : X \to [-a/3, a/3]$ such that $h(A') = \{0\}$ and $h(B') = \{1\}$. The estimate $|g(x) - h(x)| \le 2a/3$ is now direct to verify so the claim is proven.

Continuing with the proof, we apply the above claim on f in place of g to obtain a function $h_1 : X \to [-1/3, 1/3]$ such that

$$|f(x) - h_1(x)| \le 2/3 \qquad \text{for } x \in A.$$

We next apply the claim again to the map $f - h_1$, which now takes values in $[-2/3, 2/3]$ in A. This gives a map $h_2 : X \to [-2/3^2, 2/3^2]$ such that

$$|f(x) - h_1(x) - h_2(x)| \le (2/3)^2 \qquad \text{for } x \in A.$$

Continuing this process, we obtain a sequence of maps $h_k : X \to [-2^{k-1}/3^k, 2^{k-1}/3^k]$ so that the map $H_n : X \to [-1,1]$ defined by

$$H_n(x) = \sum_{k=1}^{n} h_k(x)$$

satisfies the property that $|f(x) - H_n(x)| \le (2/3)^n$ for all $x \in A$. The fact that H_n takes values in $[-1,1]$ comes from the estimate $|H_n(x)| \le \sum_{k=1}^{n} |h_k(x)| \le \sum_{k=1}^{n} 2^{k-1}/3^k = 1$.

Now for each $x \in X$, the series $\sum_{k=1}^{\infty} h_k(x)$ converges since it converges absolutely, thereby defining a function $H : X \to [-1,1]$ via $H(x) = \sum_{k=1}^{\infty} h_k(x)$. By the earlier estimates for $f - H_n$, we find that $H(x) = f(x)$ for $x \in A$. The absolute convergence of the series defining H also shows that the convergence $H_n \to H$ is uniform in X, proving also that H is continuous due to the Uniform Limit Theorem, Theorem 2.9. □

Example A.4. In this example we explore the topological space called the **long line** $\mathcal{L}$, also nicknamed the **looooooong line**. Intuitively, the long line is defined by gluing an uncountable number of half-open intervals $[0,1)$ from end-to-end, which results in a "longer" space than the usual real line $\mathbb{R}$ which is just the result of gluing a countable number of such intervals.

To define the long line properly, we first need some preliminaries about ordered sets. First of all, a total order (see Section 5.6) on a set X is called a **well-ordering** if for every subset $A \in X$, there is a least element in A, i.e. a point $a \in A$ such that $a \le b$ for all $b \in A$.

The usual order on the real line is not a well-ordering, though it is a well-ordering on the subset of positive integers. However, the *well-ordering theorem* states that every set has some total order which is also a well-ordering. This result is non-trivial and it is equivalent with the Axiom of Choice.

Let now ω_1 be the smallest **ordinal number** that, when considered as a set, is uncountable. If the reader is unfamiliar with ordinal numbers, it might help just to visualize ω_1 as $\mathbb{R}$ here since the *continuum hypothesis* asserts that the cardinality of ω_1 is the same as of the real line $\mathbb{R}$. Though it is important to note that the continuum hypothesis was shown to be unprovable by Paul Cohen in 1963. We equip ω_1 with a well-ordering simply denoted by $\leq$ here.

We next define the long line $\mathcal{L}$ as follows. We equip the set $\omega_1 \times [0, 1)$ with the **lexicographical order**, meaning that $(a, x) \leq (b, y)$ if and only if either $a < b$ (in the well-ordering on ω_1) or if $a = b$ and $x \leq y$ (in the usual order on $[0, 1)$). The long line $\mathcal{L}$ is then defined by removing the smallest element in this set and equipping the rest of the set with the order topology given by the lexicographical order. One might imagine here that this space is obtained by replacing each element of ω_1 with the half-open interval $[0, 1)$ while preserving the natural order and thus creating a "line". The long line has some curious properties as a topological space:

(a) The space $\mathcal{L}$ is first-countable, but not second-countable or Lindelöf.

(b) The space $\mathcal{L}$ is path-connected and locally homeomorphic to $\mathbb{R}$, but cannot be embedded in $\mathbb{R}$.

(c) The space $\mathcal{L}$ sequentially compact, but not compact. It is also locally compact.