

GENERAL TOPOLOGY
EXERCISES FOR SESSION 4 (WED 5.3)

Exercise 1. Show that any continuous function $f : X \rightarrow Y$ is sequentially continuous.

Exercise 2. Suppose that \mathcal{B} is a basis for the topology on Y and $f : X \rightarrow Y$ is such that $f^{-1}(B)$ is open for every $B \in \mathcal{B}$. Prove that f is continuous.

Exercise 3. Let $\chi_A : X \rightarrow \mathbb{R}$ denote the **characteristic function** of a set $A \subset X$, meaning that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Show that χ_A is discontinuous at a point x if and only if $x \in \partial A$.

Exercise 4. Let $f, g : X \rightarrow Y$ be continuous and Y be Hausdorff.

- Show that the set $\{x \in X : f(x) = y_0\}$ is closed for all $y_0 \in Y$.
- Show that the set $\{x \in X : f(x) = g(x)\}$ is closed.

Exercise 5. Show that $f : X \rightarrow Y$ is continuous if and only if for every $x \in X$ and $A \subset X$ such that $x \in \overline{A}$, we have that $f(x) \in \overline{f(A)}$.

Exercise 6. Come up with

- A continuous map which is not open.
- A continuous open map which is not closed, and vice versa.

Exercise 7. Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous at exactly one point.

Exercise 8. Show that \mathbb{R} is homeomorphic to its own subspace $(0, 1)$.

Exercise 9. Show that the pointwise limit of continuous functions does not need to be continuous (in contrast to the Uniform Limit Theorem).

Exercise 10. Let us say that $f : X \rightarrow \mathbb{R}$ is **lower semicontinuous** if the following holds: For every $a \in X$ and every $\epsilon > 0$ there exists a neighbourhood U of a such that $f(x) > f(a) - \epsilon$ for all $x \in U$. Find a topology \mathcal{T} on \mathbb{R} such that $f : X \rightarrow \mathbb{R}$ is continuous w.r.t. the topology \mathcal{T} if and only if it is lower semicontinuous.